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Probability Theory 2
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Exam preparation sheet - Solutions Part 6

0.1 Stationarity, Ergodicity, Spectral measure and spectral density

Solutions: (a) (i) X is not stationary. Since $\mathbb{E}X_n = 0$ and $\mathrm{Var}(X_n) = 1$ is constant, we have to compute the more complicated covariances. Note that by independence, $\mathrm{Cov}(X_1, X_2) = \frac{1}{\sqrt{2}}\mathrm{Cov}(\varepsilon_1, \varepsilon_1 + \varepsilon_2) = \frac{1}{\sqrt{2}}$, but $\mathrm{Cov}(X_2, X_3) = \frac{1}{\sqrt{2}\sqrt{3}}\mathrm{Cov}(\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2 + \varepsilon_3) = \frac{1}{\sqrt{6}}(\mathrm{Var}(\varepsilon_1) + \mathrm{Var}(\varepsilon_2)) = \frac{2}{\sqrt{6}}$, thus $\mathrm{Cov}(X_1, X_2) \neq \mathrm{Cov}(X_2, X_3)$ and X cannot be stationary.
(ii) We have $\mathrm{Var}(X_n) = n^2\mathrm{Var}(\varepsilon_n) = n^2$ which is not constant in n, thus X is not stationary.
(iii) Since $(\varepsilon_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence, it is stationary and ergodic. The function $g : \mathbb{R}^2 \to \mathbb{R}, g(x, y) = x^2 + y^2$ is measurable and for all $n \in \mathbb{N}_0$ it holds that $X_n = g(\varepsilon_n, \varepsilon_{n+1})$. Thus X is stationary and ergodic as a filter of a stationary and ergodic sequence.
(iv) X is stationary since for all $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ we have $\mathbb{P}^{(X_n,...,X_{n+k})} = \mathbb{P}^{(\varepsilon_0,...,\varepsilon_0)} = \mathbb{P}^{(X_0,...,X_k)}$. X is not ergodic. Proof: If X would be ergodic, the ergodic theorem would imply that $\varepsilon_0 = \frac{1}{n}\sum_{i=1}^n \varepsilon_0 = \frac{1}{n}\sum_{i=1}^n X_i \to \mathbb{E}[X_i] = \mathbb{E}\varepsilon_0 = 0$, i.e. $\varepsilon_0 = 0$. This would imply that $\mathrm{Var}(\varepsilon_0) = 0$ in contradiction to $\mathrm{Var}(\varepsilon_0) = 1$.
Remark: Another possibility is to directly argue with the shift transformation (cf. Exercise Sheet 1, Task 4(a)): Let $M \in \mathcal{B}(\mathbb{R})$ be arbitrary with $\mathbb{P}(\varepsilon_0 \in M) \notin \{0, 1\}$. Such a set M exists since ε_0 is not a.s. constant (it has positive variance). Define $A := \prod_{n\in\mathbb{N}_0} M \in \mathcal{B}(\mathbb{R}^{\mathbb{N}_0})$. The shift transformation $T : \mathbb{R}^{\mathbb{N}_0} \to \mathbb{R}^{\mathbb{N}_0}$, $(x_n)_{n\in\mathbb{N}_0} \mapsto (x_{n+1})_{n\in\mathbb{N}_0}$ satisfies $T^{-1}(A) = A$, but

$$\mathbb{P}^{(X_n)_{n\in\mathbb{N}_0}}(A) = \mathbb{P}(\forall n \in \mathbb{N}_0 : X_n \in M) = \mathbb{P}(\varepsilon_0 \in M) \notin \{0, 1\}.$$

This is a contradiction to ergodicity.
(v) X is stationary. Proof: We restrict ourselves to the case that n is odd and k is odd (all other combinations are similarly proven or trivial). In this case, it holds that $(X_n, ..., X_{n+k}) = (\varepsilon_1, \varepsilon_0, ..., \varepsilon_1, \varepsilon_0)$. It follows by independence that $\mathbb{P}^{(\varepsilon_0,\varepsilon_1)} = \mathbb{P}^{\varepsilon_0} \otimes \mathbb{P}^{\varepsilon_1} =$
$IP^{\varepsilon_1} \otimes \mathbb{P}^{\varepsilon_0} = \mathbb{P}^{(\varepsilon_1,\varepsilon_0)}$. Application of $g(x, y) = (y, x, ..., y, x)$ on $(\varepsilon_0, \varepsilon_1)$ and $(\varepsilon_1, \varepsilon_0)$ shows that also $g(\varepsilon_0, \varepsilon_1) = (\varepsilon_1, \varepsilon_0, ..., \varepsilon_1, \varepsilon_0) = (X_n, ..., X_{n+k})$ and $g(\varepsilon_1, \varepsilon_0) = (\varepsilon_0, \varepsilon_1, ..., \varepsilon_0, \varepsilon_1) = (X_0, ..., X_k)$ have the same distribution.
X is not ergodic. Proof: Let $M \in \mathcal{B}(\mathbb{R})$ be arbitrarily chosen with $\mathbb{P}(\varepsilon_0 \in M) \notin \{0, 1\}$. Such a set M exists since ε_0 is not constant a.s. Define $A := \prod_{n\in\mathbb{N}_0} M \in \mathcal{B}(\mathbb{R}^{\mathbb{N}_0})$. Obviously, the shift transformation $T : \mathbb{R}^{\mathbb{N}_0} \to \mathbb{R}^{\mathbb{N}_0}$, $(x_n)_{n\in\mathbb{N}_0} \mapsto (x_{n+1})_{n\in\mathbb{N}_0}$ fulfills $T^{-1}(A) = A$. But we have

$$\mathbb{P}^{(X_n)_{n\in\mathbb{N}_0}}(A) = \mathbb{P}(\forall n \in \mathbb{N}_0 : X_n \in M) = \mathbb{P}(\varepsilon_0 \in M, \varepsilon_1 \in M) = \mathbb{P}(\varepsilon_0 \in M)^2 \notin \{0, 1\},$$

which is a contradiction to ergodicity.
Note that

$$c(k) = \mathrm{Cov}(X_0, X_k) = \begin{cases} 1, & k \text{ even}, \\ 0, & k \text{ odd} \end{cases} = \frac{1}{2}(1 + (-1)^k).$$

does not converge to 0 for $k \to \infty$, thus $\sum_{k\in\mathbb{Z}} |c(k)| = \infty$. Thus a spectral density may not exist. We therefore directly calculate the spectral measure by making the approach $F_X =$

$a\delta_\omega + a\delta_{-\omega} + b\delta_0$ with some $a, b \geq 0$ and $\omega \in (0, \pi]$ and Dirac measures $\delta_\cdot$. The spectral measure has to fulfill

$$\frac{1}{2}(1 + (-1)^k) = c(k) = \int_{-\pi}^{\pi} e^{i\lambda k} \, \mathrm{d}F_X(\lambda) = b + a(e^{i\omega k} + e^{-i\omega k}) = b + 2a\cos(\omega k).$$

Since $\cos(\pi k) = (-1)^k$, we see that $\omega = \pi$, $b = \frac{1}{2}$ and $a = \frac{1}{4}$, i.e. $F_X = \frac{1}{4}(\delta_\pi + \delta_{-\pi}) + \frac{1}{2}$.
(vi) Since $(\varepsilon_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence, it is stationary and ergodic. The function $g : \mathbb{R}^2 \to \mathbb{R}, g(x, y) = x + \alpha y$ is measurable and for all $n \in \mathbb{N}_0$ it holds that $X_n = g(\varepsilon_n, \varepsilon_{n-1})$. Thus X is stationary and ergodic as a filter of a stationary and ergodic sequence.
We have $\mathbb{E}X_n = \mathbb{E}\varepsilon_n + \alpha\mathbb{E}\varepsilon_{n-1} = 0$, and $c(0) = \mathrm{Var}(X_n) = 1 + \alpha^2$ and $c(1) = \mathrm{Cov}(X_0, X_1) = \mathrm{Cov}(\varepsilon_0 + \alpha\varepsilon_{-1}, \varepsilon_1 + \alpha\varepsilon_0) = \alpha$. By independence of the ε_i, $c(k) = 0$ for $k \geq 2$. Thus $\sum_{k\in\mathbb{Z}} |c(k)| = |c(0)| + 2|c(1)| < \infty$, and there exists a spectral density of the form $f_X(\lambda) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} c(k)e^{i\lambda k} = \frac{1}{2\pi}(c(0) + 2c(1)\cos(\lambda)) = \frac{1}{2\pi}((1 + \alpha^2) + 2\alpha\cos(\lambda))$.
(vii) Since $(\varepsilon_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence, it is stationary and ergodic. The function $g : \mathbb{R}^3 \to \mathbb{R}, g(x, y, z) = \mathbb{1}_{\{x>z\}}$ is measurable and for all $n \in \mathbb{N}_0$ it holds that $X_n = g(\varepsilon_n, \varepsilon_{n+1}, \varepsilon_{n+2})$. Thus X is stationary and ergodic as a filter of a stationary and ergodic sequence.
We have $\mathbb{E}[X_n] = \mathbb{P}(\varepsilon_n > \varepsilon_{n+2}) = \frac{1}{2}$ by symmetry ($\varepsilon_n, \varepsilon_{n+2}$ are i.i.d.) and
$c(0) = \mathrm{Var}(X_0) = \mathbb{E}[X_0^2] - \mathbb{E}[X_0]^2 = \mathbb{P}(\varepsilon_0 > \varepsilon_2) - \mathbb{E}[X_0]^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.
$c(1) = \mathrm{Cov}(X_0, X_1) = 0$, $c(k) = \mathrm{Cov}(X_0, X_k) = 0$ for $k \geq 3$ since X_0, X_k are independent in these cases.
$\mathbb{E}[X_0X_2] = \mathbb{P}(\varepsilon_0 > \varepsilon_2 > \varepsilon_4) = \frac{1}{6}$ (by symmetry, there are 6 possible orders for the i.i.d. variables $\varepsilon_0, \varepsilon_2, \varepsilon_4$), thus $c(2) = \mathrm{Cov}(X_0, X_2) = \mathbb{E}[X_0X_2] - \mathbb{E}[X_0]^2 = \frac{1}{6} - \frac{1}{4} = -\frac{1}{12}$.
We obtain that $\sum_{k\in\mathbb{Z}} |c(k)| = |c(0)| + 2|c(2)| < \infty$, thus the spectral density exists and has the form $f_X(\lambda) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} c(k)e^{i\lambda k} = \frac{1}{2\pi}(c(0) + 2c(2)\cos(2\lambda)) = \frac{1}{2\pi}(\frac{1}{4} - \frac{1}{6}\cos(2\lambda))$.
(viii) X is not stationary since $\mathrm{Var}(X_n) = \mathrm{Var}(B_n) = n$ (recall that the Brownian motion fulfills $B_n \sim N(0, n)$), i.e. the variance of X_n is not constant in n.
(ix) Note that $D_n := B_{n+1} - B_n \overset{\text{iid}}{\sim} N(0, 1)$ for $n \in \mathbb{N}_0$. Thus $(D_n)_{n\in\mathbb{N}_0}$ is an i.i.d. sequence and thus stationary and ergodic. The function $g : \mathbb{R}^2 \to \mathbb{R}, g(x, y) = x + y$ is measurable and $X_n = g(D_n, D_{n+1})$ for all $n \in \mathbb{N}_0$. Thus X is stationary and ergodic as a filter of a stationary and ergodic sequence.
(x) X is not stationary since $\mathrm{Var}(X_{2n}) = \mathrm{Var}(B_{2n} - B_n) = n$ is not constant in n (recall that $B_{2n} - B_n \sim N(0, n)$).

(b) (i) $(U_i)_{i\in\mathbb{Z}}$ is an i.i.d. sequence and thus stationary and ergodic. The function $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = x\mathbb{1}_{\{|x| \leq a\}} + y\mathbb{1}_{\{|x|>a\}}$ is measurable and $X_i = g(U_i, U_{i-1})$ for $i \in \mathbb{Z}$, thus X is stationary as a filter of a stationary sequence.
(ii) Since $(X_i)_{i\in\mathbb{Z}}$ is stationary and ergodic, the same holds for $(|X_i|)_{i\in\mathbb{Z}}$. The ergodic theorem implies that $\frac{1}{n}\sum_{i=1}^n |X_i| \to \mathbb{E}|X_0|$ a.s. Here, we have

$$\mathbb{E}|X_0| = \underbrace{\mathbb{E}|U_0|\mathbb{1}_{\{|U_0|\leq a\}}}_{=\frac{1}{2a}\int_0^a u \, \mathrm{d}u = \frac{a}{4}} + \mathbb{E}|U_{-1}|\mathbb{E}\mathbb{1}_{\{|U_0|>a\}} = \frac{a}{4} + a \cdot \frac{1}{2} = \frac{3a}{4}.$$

(this is a finite value, which justifies the application of the ergodic theorem). We conclude that $\hat{a}_n = \frac{4}{3} \cdot \frac{1}{n}\sum_{i=1}^n |X_i| \to a$ a.s.

(c) (i) $(U_i)_{i\in\mathbb{Z}}$ is an i.i.d. sequence and thus stationary and ergodic. The function $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = \mathbb{1}_{\{x<ay\}}$ is measurable and $X_i = g(U_i, U_{i-1})$ for $i \in \mathbb{Z}$, thus X is stationary as a filter of a stationary sequence.

(ii) Since $(X_i)_{i\in\mathbb{Z}}$ is stationary and ergodic and $g:\mathbb{R}^2\to\mathbb{R}, g(x,y)=xy$ is measurable, we obtain that $Y_i := X_iX_{i-1} = g(X_i,X_{i-1})$ is stationary and ergodic. Furthermore we have $\mathbb{E}|Y_0|\le 1<\infty$ since $|X_i|\le 1$. Thus the ergodic theorem is applicable to (Y_i) and yields

$$\frac{1}{n}\sum_{i=1}^n X_iX_{i-1} = \frac{1}{n}\sum_{i=1}^n Y_i \to \mathbb{E}[Y_0] = \mathbb{E}[X_0X_1] \overset{\text{see above calc.}}{=} \frac{a^3}{6} \quad a.s.$$

Since $g(x)=(6x)^{1/3}$ is continuous, we obtain that

$$\hat{a}_n = g\Big(\frac{1}{n}\sum_{i=1}^n X_iX_{i-1}\Big) \to g(\frac{a^3}{6}) = a \quad a.s.$$

Homepage of the lecture:
`http://math.uni-heidelberg.de/stat/studinfo/teaching_stat/WT2_SS2017/index.html`