⑥ (f) $F(t) = \int_0^t \frac{1}{\sigma(s)}\,ds \quad , \quad Y_t = F(X_t), \quad dX_t = \frac{1}{2}\sigma(X_t)\sigma'(X_t)dt + \sigma(X_t)dW_t$

$$\Rightarrow dF(X_t) = \underbrace{\partial_x F}_{\substack{\text{Hauptsatz diff/Int} \\ = \frac{1}{\sigma(X_t)}}} \cdot dX_t + \frac{1}{2}\underbrace{\partial_x^2 F}_{= -\frac{\sigma'(X_t)}{\sigma(X_t)^2}} \cdot \underbrace{d\langle X\rangle_t}_{\substack{= \langle \sigma(X_t)dW_t,\, \sigma(X_t)dW_t\rangle \\ = \sigma(X_t)^2 d\langle W\rangle_t \\ = \sigma(X_t)^2 dt}}$$

$$= \frac{1}{\sigma(X_t)}\left(\frac{1}{2}\sigma(X_t)\sigma'(X_t)dt + \sigma(X_t)dW_t\right) - \frac{1}{2}\frac{\sigma'(X_t)}{\sigma(X_t)^2}\cdot\sigma(X_t)^2 dt$$

$$= \frac{1}{2}\sigma'(X_t)dt + dW_t - \frac{1}{2}\sigma'(X_t)dt$$

$$= dW_t$$

$$\Rightarrow F(X_t) - F(X_0) = W_t \;//$$

$$\Rightarrow X_t = F^{-1}(F(X_0) + W_t) \;//$$ (Lösung der SDGL)

Gegebene SDGL:

$$dX_t = \underbrace{\frac{1}{2}X_t}_{= \frac{1}{2}\sigma(X_t)\sigma'(X_t)} dt + \underbrace{\sqrt{X_t^2 - 1}}_{= \sigma(X_t)}\, dW_t$$

passt.

d.h. $\sigma(x) = \sqrt{x^2 - 1}$

$$\Rightarrow \sigma'(x) = \frac{2x}{2\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}}$$

$$\Rightarrow \sigma(x)\sigma'(x) = x \quad \checkmark$$

Damit liegt gewünschte Struktur vor.

$$\Rightarrow \underset{F}{\underset{\parallel}{F(t)}} = \int_0^t \frac{1}{\sigma(s)}ds = \int_0^t \frac{1}{\sqrt{s^2-1}}ds \overset{\text{Analysis}}{=\!=} \tanh^{-1}\left(\frac{t}{\sqrt{t^2-1}}\right)$$

$$\Leftrightarrow \tanh(F) = \frac{t}{\sqrt{t^2-1}}$$

$$\Leftrightarrow \tanh(F)^2 = \frac{t^2}{t^2-1} \Leftrightarrow \tanh(F)^2\cdot t^2 - \tanh(F)^2 - t^2 = 0$$

⑥ (f) <u>weiter:</u>

$$t^2 \cdot \left(\tanh(F)^2 - 1\right) = \tanh(F)^2$$

$$\Leftrightarrow \quad t^2 = \frac{\tanh(F)^2}{\tanh(F)^2 - 1}$$

$$\Leftrightarrow \quad t = \frac{\tanh(F)}{\sqrt{\tanh(F)^2 - 1}} = F^{-1}(F)$$

$$\rightsquigarrow \quad X_t = F^{-1}\left(F(X_0) + W_t\right) = \frac{\tanh\left(F(X_0) + W_t\right)}{\sqrt{\tanh\left(F(X_0) + W_t\right)^2 - 1}} \quad //$$

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⑥ (g)

$$X_t = X_0 \cdot \exp\Bigg(\underbrace{\int_0^t \left(\mu(s) - \tfrac{1}{2}\sigma(s)^2\right) ds}_{=: Y_t} + \underbrace{\int_0^t \sigma(s)\, dW_s}_{=: Z_t} \Bigg)$$

Also $dY_t = \left(\mu(t) - \frac{\sigma(t)^2}{2}\right) dt$

$dZ_t = \sigma(t)\, dW_t$

$$\Rightarrow \quad X_t = \underbrace{X_0 \cdot e^{Y_t + Z_t}}_{f(Y_t, Z_t)}$$

$$\underset{\text{Itô}}{\Longrightarrow} \quad dX_t = df(Y_t, Z_t)$$

$$= \underbrace{\partial_y f}_{= X_t} \cdot dY_t + \underbrace{\partial_z f}_{= X_t}\, dZ_t + \tfrac{1}{2}\partial_y^2 f \underbrace{d\langle Y\rangle_t}_{=0} + \partial_{yz}^2 f \underbrace{d\langle Y, Z\rangle_t}_{=0 \text{ da } Y \text{ beschr. Variation}}$$

$$+ \tfrac{1}{2}\underbrace{\partial_z^2 f}_{X_t} \underbrace{d\langle Z\rangle_t}_{\substack{= \langle \sigma(t) dW_t,\, \sigma(t) dW_t\rangle \\ = \sigma(t)^2 d\langle W\rangle_t \\ = \sigma(t)^2 dt}}$$

$$= X_t \underbrace{dY_t}_{= \left(\mu(t) - \frac{\sigma(t)^2}{2}\right) dt} + X_t \overbrace{dZ_t}^{= \sigma(t) dW_t} + \tfrac{1}{2} X_t \sigma(t)^2 dt$$

$$= \left[X_t \mu(t) \underbrace{- X_t \frac{\sigma(t)^2}{2} + X_t \frac{\sigma(t)^2}{2}}\right] dt + \sigma(t) X_t\, dW_t$$

$$= X_t \mu(t)\, dt + \sigma(t) X_t\, dW_t \quad //$$