Exam preparation sheet - Solutions part 5

0.5 Ito calculus

Solutions: (a) To use martingale arguments and Ito isometry, we have to show that the integrands of the stochastic integrals are in $\mathcal{L}$ (It is known from the lecture that we can assume that $W \in \mathcal{M}_2$). Define $f(s,\omega) := s^2 B_s(\omega)$. Then we have $f \in \mathcal{L}$ since

- For all $s \in [0,T]$, $f(s,\cdot) \in \mathcal{F}_s$ since $B_s \in \mathcal{F}_s$.
- For all $\omega \in \Omega$, $s \mapsto f(s,\omega)$ is continuous ($s \mapsto B_s$ is continuous) and thus (together with the first point) f is product-measurable.
- We have
$$\int_0^T \mathbb{E}[f(s,\cdot)^2]\ \mathrm{d}s = \int_0^T s^4 \underbrace{\mathbb{E}[B_s^2]}_{=s}\ \mathrm{d}s = \frac{1}{6}T^6 < \infty. \quad (*)$$

In the same manner we obtain that $f(s,\omega) := B_s(\omega)^2$ fulfills $g \in \mathcal{L}$ since

- For all $s \in [0,T]$, $g(s,\cdot) \in \mathcal{F}_s$ since $B_s \in \mathcal{F}_s$.
- For all $\omega \in \Omega$, $s \mapsto g(s,\omega)$ is continuous ($s \mapsto B_s$ is continuous) and thus (together with the first point) g is product-measurable.
- We have by the hint $\mathbb{E}[Z^4] = 3$ for $Z \sim N(0,1)$, thus
$$\int_0^T \mathbb{E}[g(s,\cdot)^2]\ \mathrm{d}s = \int_0^T \underbrace{\mathbb{E}[B_s^4]}_{=3s^2}\ \mathrm{d}s = T^3 < \infty. \quad (**)$$

Since $f,g \in \mathcal{L}$, we have that $(X_t)_{t\geq 0} = \big(\int_0^t f(s,\cdot)\ \mathrm{d}B_s\big)_{t\geq 0}$ and $\big(\int_0^t g(s,\omega)\ \mathrm{d}B_s\big)_{t\geq 0}$ are martingales w.r.t. $\mathcal{F}$. We obtain that for all $0 \leq t \leq T$,

$$\mathbb{E}X_t = \mathbb{E}\underbrace{X_0}_{=0} = 0, \qquad \mathbb{E}Y_t = e^{\lambda t}\mathbb{E}\Big[\int_0^t g(s,\omega)\ \mathrm{d}B_s\Big] = e^{\lambda t}\mathbb{E}\Big[\underbrace{\int_0^0 g(s,\omega)\ \mathrm{d}B_s}_{=0}\Big] = 0.$$

By the Ito isometry, we obtain that

$$\mathbb{E}[X_t^2] = \mathbb{E}\Big[\Big(\int_0^t f(s,\cdot)\ \mathrm{d}B_s\Big)^2\Big] = \int_0^t \mathbb{E}[f(s,\cdot)^2]\ \mathrm{d}s \overset{(*)}{=} \frac{t^6}{6},$$

and

$$\mathbb{E}[Y_t^2] = e^{2\lambda t}\mathbb{E}\Big[\Big(\int_0^t g(s,\cdot)\ \mathrm{d}B_s\Big)^2\Big] = e^{2\lambda t}\int_0^t \mathbb{E}[g(s,\cdot)^2]\ \mathrm{d}s \overset{(**)}{=} e^{2\lambda t}t^3,$$

and

$$\mathbb{E}[X_tY_t] = e^{\lambda t}\mathbb{E}\Big[\int_0^t f(s,\cdot)\ \mathrm{d}B_s \cdot \int_0^t g(s,\cdot)\ \mathrm{d}B_s\Big] = e^{\lambda t}\int_0^t \mathbb{E}[\underbrace{f(s,\cdot)g(s,\cdot)}_{=s^2B_s^3}]\ \mathrm{d}s \overset{\mathbb{E}[B_s^3]=0}{=} 0.$$

(b) We first show that $f(s,\omega) := e^{\lambda B_s(\omega)}$ is in $\mathcal{L}$:

- For all $s \in [0,T]$, $f(s,\cdot) \in \mathcal{F}_s$ since $B_s \in \mathcal{F}_s$.

- For all $\omega \in \Omega$, $s \mapsto f(s,\omega)$ is continuous ($s \mapsto B_s$ is continuous) and thus (together with the first point) f is product-measurable.

- We have by the hint $\mathbb{E}[e^{\sigma Z}] = e^{\sigma^2/2}$ for $Z \sim N(0,1)$, thus

$$\int_0^T \mathbb{E}[f(s,\cdot)^2]\ \mathrm{d}s = \int_0^T \underbrace{\mathbb{E}[e^{2\lambda B_s}]}_{=e^{2\lambda^2 s}}\ \mathrm{d}s = \Big[\frac{e^{2\lambda^2 s}}{2\lambda^2}\Big]_0^T = \frac{1}{2\lambda^2}\big(e^{2\lambda^2 T} - 1\big) < \infty. \quad (***)$$

Since $f \in \mathcal{L}$, we have that $\big(\int_0^t f(s,\cdot)\ \mathrm{d}B_s\big)_{t\geq 0}$ is a martingale. We obtain that

$$\mathbb{E}X_t = 1 + \mathbb{E}\Big[\int_0^t f(s,\cdot)\ \mathrm{d}B_s\Big] = 1 + \underbrace{\mathbb{E}\Big[\int_0^0 f(s,\cdot)\ \mathrm{d}B_s\Big]}_{=0} = 1.$$

By Ito's isometry, we have

$$\begin{aligned} \mathbb{E}[X_t^2] &\overset{\mathbb{E}[Y^2]=\mathrm{Var}(Y)+\mathbb{E}[Y]^2}{=} \mathbb{E}[(X_t-1)^2] + 1 = 1 + \mathbb{E}\Big[\Big(\int_0^t f(s,\cdot)\ \mathrm{d}B_s\Big)^2\Big] = 1 + \int_0^t \mathbb{E}[f(s,\cdot)^2]\ \mathrm{d}s \\ &\overset{(***)}{=} 1 + \frac{1}{2\lambda^2}\big(e^{2\lambda^2 t} - 1\big). \end{aligned}$$

(c) We use the property of stochastic integrals $\int_0^t f(s,\cdot)\ \mathrm{d}B_s$ to be martingales if the integrand $f \in \mathcal{L}$.
(i) Note that $X_t = g(t,B_t)$ with $g(t,x) := 5 + x^4 - 6tx^2 + 3t^2$. By Ito's formula ($\partial_t g(t,x) = -6x^2 + 6t$, $\partial_x g(t,x) = 4x^3 - 12tx$, $\partial_x^2 g(t,x) = 12x^2 - 12t$) we obtain (note that $\partial_t g + \frac{1}{2}\partial_x^2 g = 0$):

$$\begin{aligned} \mathrm{d}X_t &= \big[(\partial_t g)(t,B_t) + \frac{1}{2}\partial_x^2 g(t,B_t)\big]\ \mathrm{d}t + \partial_x g(t,B_t)\ \mathrm{d}B_t \\ &= \big(4B_t^3 - 12tB_t\big)\ \mathrm{d}B_t, \end{aligned}$$

or equivalently for $0 \leq t \leq T$ (we have $X_0 = 5$):

$$X_t = 5 + \int_0^t \big(4B_s^3 - 12sB_s\big)\ \mathrm{d}B_s.$$

Define $f(s,\omega) := 4B_s(\omega)^3 - 12sB_s(\omega)$. We now show that $f \in \mathcal{L}$:

- For all $s \in [0,T]$, $f(s,\cdot) \in \mathcal{F}_s$ since $B_s \in \mathcal{F}_s$.

- For all $\omega \in \Omega$, $s \mapsto f(s,\omega)$ is continuous ($s \mapsto B_s$ is continuous) and thus (together with the first point) f is product-measurable.

- We have $(B_s^3 - 3sB_s)^2 = B_s^6 - 6sB_s^4 + 9s^2B_s^2$. Since $\mathbb{E}[Z^6] = 15$ and $\mathbb{E}[Z^4] = 3$ for $Z \sim N(0,1)$, we have

$$\begin{aligned} \int_0^T \mathbb{E}[f(s,\cdot)^2]\ \mathrm{d}s &= 4^2 \cdot \Big[\int_0^T \underbrace{\mathbb{E}[B_s^6]}_{=15s^3}\ \mathrm{d}s - 6\int_0^T s\,\underbrace{\mathbb{E}[B_s^4]}_{=3s^2}\ \mathrm{d}s + 9\int_0^T s^2\,\underbrace{\mathbb{E}[B_s^2]}_{=s}\ \mathrm{d}s\Big] \\ &= 4^2 \cdot 6 \cdot \int_0^T s^3\ \mathrm{d}s = 24T^4 < \infty \end{aligned}$$

(such an exact calculation is not necessary, one could also bound the integral by some finite term from above!)

We conclude that $\left(\int_0^t f(s,\cdot)\ \mathrm{d}B_s\right)_{t\in[0,T]}$ is a martingale, thus $(X_t)_{t\in[0,T]}$ is a martingale.
(ii) Note that $X_t = g(t,B_t)$ with $g(t,x) := e^{\lambda^2 t/2}\sin(\lambda x)$. By Ito's formula ($\partial_t g(t,x) = \frac{\lambda^2}{2} g(t,x)$, $\partial_x g(t,x) = \lambda e^{\lambda^2 t/2}\cos(\lambda x)$, $\partial_x^2 g(t,x) = -\lambda^2 g(t,x)$) we obtain (note that $\partial_t g + \frac{1}{2}\partial_x^2 g = 0$):

$$\begin{aligned}\mathrm{d}X_t &= \left[(\partial_t g)(t,B_t) + \frac{1}{2}\partial_x^2 g(t,B_t)\right]\ \mathrm{d}t + \partial_x g(t,B_t)\ \mathrm{d}B_t \\ &= \left(\lambda e^{\lambda^2 t/2}\cos(\lambda B_t)\right)\ \mathrm{d}B_t,\end{aligned}$$

or equivalently for $0 \le t \le T$ (we have $X_0 = 0$):

$$X_t = \int_0^t \left(\lambda e^{\lambda^2 s/2}\cos(\lambda B_s)\right)\ \mathrm{d}B_s.$$

Define $f(s,\omega) := \lambda e^{\lambda^2 s/2}\cos(\lambda B_s(\omega))$. We now show that $f \in \mathcal{L}$:

- For all $s \in [0,T]$, $f(s,\cdot) \in \mathcal{F}_s$ since $B_s \in \mathcal{F}_s$.
- For all $\omega \in \Omega$, $s \mapsto f(s,\omega)$ is continuous ($s \mapsto B_s$ is continuous) and thus (together with the first point) f is product-measurable.
- We have

$$\begin{aligned}\int_0^T \mathbb{E}[f(s,\cdot)^2]\ \mathrm{d}s &= \int_0^T \lambda^2 e^{\lambda^2 s}\mathbb{E}[\cos(\lambda B_s)^2]\ \mathrm{d}s \overset{|\cos(\cdot)|\le 1}{\le} \lambda^2 \int_0^T e^{\lambda^2 s}\ \mathrm{d}s \\ &= e^{\lambda^2 T} - 1 < \infty\end{aligned}$$

We conclude that $(X_t)_{t\in[0,T]} = \left(\int_0^t f(s,\cdot)\ \mathrm{d}B_s\right)_{t\in[0,T]}$ is a martingale.
(iii) Note that $X_t = g(t,B_t)$ with $g(t,x) := e^{-\lambda^2 t/2}\cosh(\lambda x)$. By Ito's formula ($\partial_t g(t,x) = -\frac{\lambda^2}{2} g(t,x)$, $\partial_x g(t,x) = \lambda e^{-\lambda^2 t/2}\sinh(\lambda x)$, $\partial_x^2 g(t,x) = \lambda^2 g(t,x)$) we obtain (note that $\partial_t g + \frac{1}{2}\partial_x^2 g = 0$):

$$\begin{aligned}\mathrm{d}X_t &= \left[(\partial_t g)(t,B_t) + \frac{1}{2}\partial_x^2 g(t,B_t)\right]\ \mathrm{d}t + \partial_x g(t,B_t)\ \mathrm{d}B_t \\ &= \left(\lambda e^{-\lambda^2 t/2}\sinh(\lambda B_t)\right)\ \mathrm{d}B_t,\end{aligned}$$

or equivalently for $0 \le t \le T$ (we have $X_0 = 1$):

$$X_t = 1 + \int_0^t \left(\lambda e^{-\lambda^2 s/2}\sinh(\lambda B_s)\right)\ \mathrm{d}B_s.$$

Define $f(s,\omega) := \lambda e^{-\lambda^2 s/2}\sinh(\lambda B_s(\omega))$. We now show that $f \in \mathcal{L}$:

- For all $s \in [0,T]$, $f(s,\cdot) \in \mathcal{F}_s$ since $B_s \in \mathcal{F}_s$.
- For all $\omega \in \Omega$, $s \mapsto f(s,\omega)$ is continuous ($s \mapsto B_s$ is continuous) and thus (together with the first point) f is product-measurable.
- We have (note that $\sinh(x)^2 = \frac{e^{2x}+e^{-2x}}{4} - \frac{1}{2} \le \frac{e^{2x}+e^{-2x}}{4}$ and $\mathbb{E}[e^{\sigma Z}] = e^{\sigma^2/2}$ for $Z \sim N(0,1)$)

$$\begin{aligned}\int_0^T \mathbb{E}[f(s,\cdot)^2]\ \mathrm{d}s &= \lambda^2 \int_0^T e^{-\lambda^2 s}\mathbb{E}[\underbrace{\sinh(\lambda B_s)^2}_{\le \frac{1}{4}(e^{2\lambda B_s}+e^{-2\lambda B_s})}]\ \mathrm{d}s \le \frac{\lambda^2}{4}\int_0^T e^{-\lambda^2 s}\cdot 2e^{2\lambda^2 s}\ \mathrm{d}s \\ &= \frac{\lambda^2}{2}\int_0^T e^{\lambda^2 s}\ \mathrm{d}s = \frac{1}{2}\left[e^{\lambda^2 T} - 1\right] < \infty.\end{aligned}$$

We conclude that $\big(\int_0^t f(s,\cdot)\,\mathrm{d}B_s\big)_{t\in[0,T]}$ is a martingale. Thus $(X_t)_{t\in[0,T]} = \big(1+\int_0^t f(s,\cdot)\,\mathrm{d}B_s\big)_{t\in[0,T]}$ is a martingale.

(iv) Define $Y_t = g(t,B_t)$ with $g(t,x) := t^2x^3$. By Ito's formula ($\partial_t g(t,x) = 2tx^3$, $\partial_x g(t,x) = 3t^2x^2$, $\partial_x^2 g(t,x) = 6t^2x$) we obtain:

$$\begin{aligned}\mathrm{d}Y_t &= \big[(\partial_t g)(t,B_t) + \frac{1}{2}\partial_x^2 g(t,B_t)\big]\ \mathrm{d}t + \partial_x g(t,B_t)\ \mathrm{d}B_t\\ &= \big(2tB_t^3 + 3t^2B_t\big)\ \mathrm{d}t + \big(3t^2B_t^2\big)\ \mathrm{d}B_t,\end{aligned}$$

or equivalently for $0\le t\le T$ (we have $Y_0 = 0$):

$$Y_t = \int_0^t sB_s(2B_s^2+3s)\ \mathrm{d}s + \int_0^t \big(3s^2B_s^2\big)\ \mathrm{d}B_s,$$

i.e. $X_t = Y_t - \int_0^t sB_s(B_s^2+3s)\ \mathrm{d}s = \int_0^t \big(3s^2B_s^2\big)\ \mathrm{d}B_s$.

Define $f(s,\omega) := 3s^2B_s(\omega)^2$. We now show that $f\in\mathcal{L}$:

- For all $s\in[0,T]$, $f(s,\cdot)\in\mathcal{F}_s$ since $B_s\in\mathcal{F}_s$.
- For all $\omega\in\Omega$, $s\mapsto f(s,\omega)$ is continuous ($s\mapsto B_s$ is continuous) and thus (together with the first point) f is product-measurable.
- We have (note that $\mathbb{E}[Z^4] = 3$ for $Z\sim N(0,1)$):

$$\int_0^T \mathbb{E}[f(s,\cdot)^2]\ \mathrm{d}s \;=\; 9\int_0^T s^4\underbrace{\mathbb{E}[B_s^4]}_{=3s^2}\ \mathrm{d}s = 27\int_0^T s^6\ \mathrm{d}s = \frac{27}{7}T^7 < \infty.$$

We conclude that $(X_t)_{t\in[0,T]} = \big(\int_0^t f(s,\cdot)\ \mathrm{d}B_s\big)_{t\in[0,T]}$ is a martingale.

(d) (i) Let $T := t$. We search for g such that $\partial_x g(t,x) = e^{-x}$. This is fulfilled for $g(t,x) = -e^{-x}$. Put $Y_t := g(t,B_t) = -e^{-B_t}$. By Ito's formula, we have

$$\begin{aligned}\mathrm{d}Y_t &= \big[(\partial_t g)(t,B_t) + \frac{1}{2}\partial_x^2 g(t,B_t)\big]\ \mathrm{d}t + \partial_x g(t,B_t)\ \mathrm{d}B_t\\ &= -\frac{1}{2}e^{-B_t}\ \mathrm{d}t + e^{-B_t}\ \mathrm{d}B_t,\end{aligned}$$

or equivalently (since $Y_0 = -1$),

$$-e^{-B_t} = -1 - \frac{1}{2}\int_0^t e^{-B_s}\ \mathrm{d}s + \int_0^t e^{-B_s}\ \mathrm{d}B_s \quad\Leftrightarrow\quad \int_0^t e^{-B_s}\ \mathrm{d}B_s = 1 + \frac{1}{2}\int_0^t e^{-B_s}\ \mathrm{d}s - e^{-B_t}.$$

We obtain that

$$e^{B_t}\int_0^t e^{-B_s}\ \mathrm{d}B_s = e^{B_t} + \frac{1}{2}\int_0^t e^{B_t-B_s}\ \mathrm{d}s - 1$$

Since $\mathbb{E}[e^Z] = e^{\sigma^2/2}$ for $Z\sim N(0,\sigma^2)$, we have with Fubini's theorem (the integrand is positive)

$$\begin{aligned}\mathbb{E}\Big[e^{B_t}\int_0^t e^{-B_s}\ \mathrm{d}B_s\Big] &= \mathbb{E}[e^{B_t}] + \frac{1}{2}\int_0^t \mathbb{E}\big[e^{B_t-B_s}\big]\ \mathrm{d}s - 1\\ &= e^{t/2} + \frac{1}{2}\underbrace{\int_0^t e^{(t-s)/2}\ \mathrm{d}s}_{=[-2e^{(t-s)/2}]_0^t = 2(e^{t/2}-1)} - 1\\ &= e^{t/2} + (e^{t/2}-1) - 1\\ &= 2(e^{t/2}-1).\end{aligned}$$

(ii) Let $T := t$. We have $B_t = \int_0^t 1\ \mathrm{d}B_s$ (obviously $1\in\mathcal{L}$), and $f(s,\omega) := B_s(\omega)^2$ fulfills $f\in\mathcal{L}$ since

- For all $s \in [0,T]$, $f(s,\cdot) \in \mathcal{F}_s$ since $B_s \in \mathcal{F}_s$.
- For all $\omega \in \Omega$, $s \mapsto f(s,\omega)$ is continuous ($s \mapsto B_s$ is continuous) and thus (together with the first point) f is product-measurable.
- We have (since $\mathbb{E}[Z^4] = 3$ for $Z \sim N(0,1)$):

$$\int_0^T \mathbb{E}[f(s,\cdot)^2]\ \mathrm{d}s = \int_0^T \underbrace{\mathbb{E}[B_s^4]}_{=3s^2}\ \mathrm{d}s = T^3 < \infty.$$

By Ito's isometry, we have

$$\mathbb{E}\big[B_t \cdot \int_0^t B_s^2\ \mathrm{d}B_s\big] = \mathbb{E}\big[\int_0^t 1\ \mathrm{d}B_s \cdot \int_0^t f(s,\cdot)\ \mathrm{d}B_s\big] = \int_0^t \mathbb{E}[1 \cdot f(s,\cdot)]\ \mathrm{d}s = \int_0^t \underbrace{\mathbb{E}[B_s^2]}_{=s}\ \mathrm{d}s = \frac{1}{2}t^2.$$

(e) Define $g(t,x) := f(t)x$ and $Y_t := g(t,B_t) = f(t)B_t$. Obviously, g is twice differentiable. By Ito's formula we obtain for $t \in [0,T]$ a.s.:

$$\begin{aligned} \mathrm{d}Y_t &= \big[(\partial_t g)(t,B_t) + \frac{1}{2}\partial_x^2 g(t,B_t)\big]\ \mathrm{d}t + \partial_x g(t,B_t)\ \mathrm{d}B_t \\ &= f'(t)B_t\ \mathrm{d}t + f(t)\ \mathrm{d}B_t. \end{aligned}$$

Since $Y_0 = f(0)B_0 = 0$, this is equivalent to: For all $t \in [0,T]$ it holds a.s. that

$$f(t)B_t = Y_t = \int_0^t f'(s)B_s\ \mathrm{d}s + \int_0^t f(s)\ \mathrm{d}B_s \quad \Leftrightarrow \quad f(t)B_t - \int_0^t B_s f'(s)\ \mathrm{d}s = \int_0^t f(s)\ \mathrm{d}B_s.$$