Exam preparation sheet - Solutions part 3

0.4 Weak convergence in $C[0,1]$

Solutions: For (a) - (e) we use the convergence principle from the lecture. This means for $X_n \overset{D}{\to} X$ in $(C[0,1], \|\cdot\|_\infty)$ we have to show that

(C1) There exist $\gamma, \kappa > 0$, $K \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that

$$\forall s,t \in [0,1], n \geq n_0, \varepsilon > 0 : \qquad \mathbb{P}(|X_n(s) - X_n(t)| \geq \varepsilon) \leq \frac{K|s-t|^{1+\gamma}}{\varepsilon^\kappa}.$$

(C2) The finite-dimensional distributions converge, i.e. for all $k \in \mathbb{N}$, for all $0 \leq t_1 < ... < t_k \leq 1$, $(X_n(t_1), ..., X_n(t_k)) \overset{D}{\to} (X(t_1), ..., X(t_k))$.

(a) Since $X^{(n)}$ is a Gaussian process, we have that $X^{(n)}(s) - X^{(n)}(t) \sim N(a_n, \tau_n^2)$ with

$$\begin{aligned}
a_n &= \mathbb{E}[X^{(n)}(s) - X^{(n)}(t)] = \mu_n(t) - \mu_n(s) = \frac{1}{n}\big(\sin(t) - \sin(s)\big) \\
\tau_n^2 &= \mathrm{Var}(X^{(n)}(s) - X^{(n)}(t)) = \mathrm{Var}(X^{(n)}(s)) + \mathrm{Var}(X^{(n)}(t)) - 2\mathrm{Cov}(X^{(n)}(s), X^{(n)}(t)) \\
&= \big[s - \frac{1}{2\sqrt{n}}\big] + \big[t - \frac{1}{2\sqrt{n}}\big] - \big[s + t - \sqrt{(s-t)^2 + \frac{1}{n}}\big] \\
&= \sqrt{(s-t)^2 + \frac{1}{n}} - \frac{1}{\sqrt{n}} \overset{\sqrt{x}-\sqrt{y}=\frac{x-y}{\sqrt{x}+\sqrt{y}}}{=} \frac{(s-t)^2}{\sqrt{(s-t)^2 + \frac{1}{n}} + \frac{1}{\sqrt{n}}} \leq \frac{(s-t)^2}{\sqrt{(s-t)^2}} = |s-t|
\end{aligned}$$

With the hint and $|\sin(s) - \sin(t)| \leq |s-t|$ we conclude that

$$\mathbb{E}\big[|X^{(n)}(s) - X^{(n)}(t)|^4\big] = a_n^4 + 6a_n^2\tau_n^2 + 3\tau_n^4 \leq \frac{1}{n^4}|s-t|^4 + \frac{6}{n^2}|s-t|^2 \cdot |s-t| + 3|s-t|^2$$

Since $|s-t| \leq 1$ and $n \geq 1$, we obtain that $\mathbb{E}\big[|X^{(n)}(s) - X^{(n)}(t)|^4\big] \leq 10|s-t|^2$. We conclude with Markov's inequality that

$$\mathbb{P}(|X^{(n)}(s) - X^{(n)}(t)| \geq \varepsilon) \leq \frac{\mathbb{E}\big[|X^{(n)}(s) - X^{(n)}(t)|^4\big]}{\varepsilon^4} \leq \frac{10|s-t|^2}{\varepsilon^4},$$

i.e. (C1) is satisfied.
Now let $k \in \mathbb{N}$, $0 \leq t_1 < ... < t_k \leq 1$. Since $X^{(n)}$ is a Gaussian process, $(X^{(n)}(t_1), ..., X^{(n)}(t_k)) \sim N(m_n, \Sigma_n)$ with $m_n \in \mathbb{R}^k, \Sigma_n \in \mathbb{R}^{k \times k}$. Here,

$$\begin{aligned}
m_n &= \begin{pmatrix} \mu_n(t_1) \\ \vdots \\ \mu_n(t_k) \end{pmatrix} = \frac{1}{n}\begin{pmatrix} \sin(t_1) \\ \vdots \\ \sin(t_k) \end{pmatrix} \to 0, \\
\Sigma_{n,ij} &= \mathrm{Cov}(X^{(n)}(t_i), X^{(n)}(t_j)) = \frac{1}{2}(t_i + t_j - \sqrt{(t_i - t_j)^2 + \frac{1}{n}}) \to \frac{1}{2}(t_i + t_j - |t_i - t_j|) = \min\{t_i, t_j\}.
\end{aligned}$$

Since normal distributions converge if the parameters converge, we have shown that

$$(X^{(n)}(t_1), ..., X^{(n)}(t_k)) \overset{D}{\to} N(m, \Sigma)$$

with $m = 0$ and $\Sigma_{ij} = \min\{t_i, t_j\}$ which is exactly the distribution of $(B_{t_1}, ..., B_{t_k})$, where B is a Brownian motion. Thus we have shown that

$$(X^{(n)}(t_1), ..., X^{(n)}(t_k)) \overset{D}{\to} (B_{t_1}, ..., B_{t_k}).$$

This shows (C2) and we conclude that $X^{(n)} \xrightarrow{D} B$ in $(C[0,1], \|\cdot\|_\infty)$.

(b) (i) Note that $\mathbb{E}\varepsilon_i = 0$ and thus $\mathbb{E}X_n(a) = 0$. We conclude that

$$\begin{aligned}\mathbb{E}[|X_n(a) - X_n(a')|^2] &= \operatorname{Var}(X_n(a) - X_n(a')) \overset{k=0 \text{ vanishes}}{=} \operatorname{Var}\Big(\sum_{k=1}^{n}(a^k - (a')^k)\varepsilon_{n-k}\Big) \\ &\overset{\varepsilon_i \text{ iid}}{=} \sum_{k=1}^{n} \operatorname{Var}((a^k - (a')^k)\varepsilon_{n-k}) = \sigma^2 \sum_{k=1}^{n}(a^k - (a')^k)^2.\end{aligned}$$

By Taylor's formula applied to $f(x) = x^k$, we have $f(x) - f(x') = (x - x')f'(\tilde{x})$ with some $\tilde{x}$ between x, x'. In the above setting, we obtain for $(a^k - (a')^k)^2 = \big((a - a')k\tilde{a}^{k-1}\big)^2 \leq (a - a')^2 k^2 (1-\delta)^{2(k-1)}$ since $\tilde{a}$ is between a, a' which are both in A. Thus by using $\sum_{k=1}^{n} \leq \sum_{k=1}^{\infty}$,

$$\mathbb{E}[|X_n(a) - X_n(a')|^2] \leq (a - a')^2 \underbrace{\sigma^2 \sum_{k=1}^{\infty} k^2 (1-\delta)^{2(k-1)}}_{=:C},$$

where the sum converges (and is $< \infty$) for instance by quotient criterium.
(ii) By (i), we have for all $\varepsilon > 0$ that for all $a, a' \in A$:

$$\mathbb{P}(|X_n(a) - X_n(a')| \geq \varepsilon) \leq \frac{\mathbb{E}[|X_n(a) - X_n(a')|^2]}{\varepsilon^2} \leq \frac{C}{\varepsilon^2}|a - a'|^2,$$

i.e. (C1) is fulfilled with $\gamma = 1$.
For $l \in \mathbb{N}$, $-1 + \delta \leq a_1 < ... < a_l \leq 1 - \delta$, we have

$$\begin{pmatrix} X_n(a_1) \\ \vdots \\ X_n(a_l) \end{pmatrix} = \sum_{k=0}^{n} \varepsilon_{n-k} \cdot \begin{pmatrix} a_1^k \\ \vdots \\ a_l^k \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & a_1 & \dots & a_1^n \\ 1 & a_2 & \dots & a_2^n \\ \vdots & \vdots & & \vdots \\ 1 & a_l & \dots & a_l^n \end{pmatrix}}_{=:B} \cdot \underbrace{\begin{pmatrix} \varepsilon_n \\ \vdots \\ \varepsilon_0 \end{pmatrix}}_{\sim N(0, \sigma^2 I_{l \times l})} \sim N(0, \underbrace{\sigma^2 BB'}_{=:\Sigma^{(n)}}).$$

(B is the Vandermonde matrix and thus has full rank since the a_i are all different). We have

$$\begin{aligned}\Sigma_{ij}^{(n)} &= \operatorname{Cov}(X_n(a_i), X_n(a_j)) = \operatorname{Cov}\Big(\sum_{k=0}^{n} a_i^k \varepsilon_{n-k}, \sum_{k=0}^{n} a_j^k\ \varepsilon_{n-k}\Big) \overset{\varepsilon_i \text{ iid}}{=} \sigma^2 \sum_{k=0}^{n} (a_i a_j)^k \\ &= \frac{1 - (a_i a_j)^{n+1}}{1 - a_i a_j} \to \frac{\sigma^2}{1 - a_i a_j}.\end{aligned}$$

This shows that $\Sigma^{(n)}$ converges to Σ with $\Sigma_{ij} = \frac{\sigma^2}{1 - a_i a_j}$. Since Gaussian distributions converge if the parameters converge, we obtain that $(X_n(a_1), ..., X_n(a_l)) \xrightarrow{D} N(0, \Sigma)$. Since Z is a centered Gaussian process with covariance function $\operatorname{Cov}(Z_a, Z_{a'}) = \frac{\sigma^2}{1 - aa'}$, we have that $(Z_{a_1}, ..., Z_{a_l}) \sim N(0, \Sigma)$ with the same Σ as before. Thus we have shown that

$$(X_n(a_1), ..., X_n(a_l)) \xrightarrow{D} (Z_{a_1}, ..., Z_{a_l}),$$

i.e. (C2). By the convergence principle (applied to $C(A)$ as the hint suggests), we have $X_n \xrightarrow{D} Z$ in $(C(A), \|\cdot\|_\infty)$.
(iii) By the hint, we define $\Phi : (C(A) \times A, \|\cdot\|_\infty \times |\cdot|) \to (\mathbb{R}, |\cdot|)$, $\Phi(f, x) := f(x)$. Φ is continuous,

since for $\|f_n - f\|_\infty \to 0$ and $|x_n - x| \to 0$, we have $|\Phi(f_n, x_n) - \Phi(f, x)| = |f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \leq \|f_n - f\|_\infty + |f(x_n) - f(x)| \to 0$ since f is continuous and $x_n \to x$.
Since $T_n \xrightarrow{\mathbb{P}} \tau$ and τ is a deterministic constant, we have by a theorem from the lecture and (ii) that $(X_n, T_n) \xrightarrow{D} (Z, \tau)$ in $(C(A) \times A, \|\cdot\|_\infty \times |\cdot|)$.
By the continuous mapping theorem, we obtain that $X_n(T_n) = \Phi(X_n, T_n) \xrightarrow{D} \Phi(Z, \tau) = Z_\tau \sim N(0, \frac{\sigma^2}{1-\tau^2})$ (Z is a centered Gaussian process, so Z_τ is normal distributed with mean 0 and variance $\mathrm{Var}(Z_\tau) = \mathrm{Cov}(Z_\tau, Z_\tau) = \frac{\sigma^2}{1-\tau^2}$).

(c) (i) Note that $\mathbb{E}\hat{E}_n(M) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}[X_i h_M(X_i)] = \mathbb{E}[X_1 h_M(X_1)] = E(M)$ since X_i are iid, thus $\mathbb{E}\hat{P}_n(M) = 0$. We conclude (note that $\mathrm{Var}(Z) = \mathrm{Var}(Z + c)$ for constants c) that

$$\begin{aligned}
& \mathbb{E}\big[|\hat{P}_n(M) - \hat{P}_n(M')|^2\big] \\
= \quad & \mathrm{Var}(\hat{P}_n(M) - \hat{P}_n(M')) = \mathrm{Var}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^n \big(X_i h_M(X_i) - X_i h_{M'}(X_i)\big)\Big) \\
\overset{X_i \text{ iid}}{=} \quad & \frac{1}{n}\sum_{i=1}^n \mathrm{Var}(X_i h_M(X_i) - X_i h_{M'}(X_i)) = \mathrm{Var}(X_1 h_M(X_1) - X_1 h_{M'}(X_1)) \\
\overset{\mathrm{Var}(Z)=\mathbb{E}[Z^2]-\mathbb{E}[Z]^2}{\leq} \quad & \mathbb{E}[X_1^2 |h_M(X_1) - h_{M'}(X_1)|^2].
\end{aligned}$$

It is easy to see that $h_M(x) - h_{M'}(x) \leq \frac{1}{L} \cdot |M - M'|$ independent of x. Proof: w.l.o.g. assume that $M < M'$. It holds that

$$h_M(x) = \begin{cases} 1, & x \in [-M, M], \\ 0, & x \in [-(M+L), (M+L)], \\ \frac{1}{L}(x + (M+L)), & x \in [-(M+L), M], \\ \frac{1}{L}((M+L) - x), & x \in [M, M+L]. \end{cases}$$

w.l.o.g. assume that $x \in [M, M+L]$ (the other cases are easier). If $x \in [M', M'+L]$, then $|h_M(x) - h_{M'}(x)| = \frac{1}{L}|(M + L - x) - (M' + L - x)| = \frac{1}{L}|M - M'|$. If $x \in [M, M']$, then $|h_M(x) - h_{M'}(x)| = |\frac{1}{L}(M + L - x) - 1| = \frac{1}{L}|M - x| \leq \frac{1}{L}|M - M'|$.

Thus we have

$$\mathbb{E}\big[|\hat{P}_n(M) - \hat{P}_n(M')|^2\big] \leq \frac{\mathbb{E}[X_1^2]}{L^2}|M - M'|^2.$$

(ii) By (i), we have for all $\varepsilon > 0$ that for all $M, M' \in [0, 1]$:

$$\mathbb{P}(|\hat{P}_n(M) - \hat{P}_n(M')| \geq \varepsilon) \leq \frac{\mathbb{E}[|\hat{P}_n(M) - \hat{P}_n(M')|^2]}{\varepsilon^2} \leq \frac{\mathbb{E}[X_1^2]}{L^2\varepsilon^2}|M - M'|^2,$$

i.e. (C1) is fulfilled with $\gamma = 1$.
Now let $k \in \mathbb{N}$, $0 \leq M_1 < ... < M_k \leq 1$. By the multivariate central limit theorem, we have

$$\begin{pmatrix} \hat{P}_n(M_1) \\ \vdots \\ \hat{P}_n(M_k) \end{pmatrix} = \frac{1}{\sqrt{n}}\sum_{i=1}^n \begin{pmatrix} X_i h_{M_1}(X_i) - \mathbb{E}[X_i h_{M_1}(X_i)] \\ \vdots \\ X_i h_{M_k}(X_i) - \mathbb{E}[X_i h_{M_k}(X_i)] \end{pmatrix} \xrightarrow{D} N(0,\ \Sigma),$$

where $\Sigma_{ij} = \mathrm{Cov}(X_1 h_{M_i}(X_1), X_1 h_{M_j}(X_1))$. So if we ask Z to have the covariance function $\gamma(M, M') = \mathbb{E}[Z_M Z_{M'}] = \mathrm{Cov}(X_1 h_M(X_1), X_1 h_{M'}(X_1))$, then $(Z_{M_1}, ..., Z_{M_k}) \sim N(0, \Sigma)$ (multivariate normal with mean 0 since Z is a centered Gaussian process). Thus with this choice of

$\gamma(M, M')$ we have shown that

$$(\hat{P}_n(M_1), ..., \hat{P}_n(M_k)) \xrightarrow{D} (Z_{M_1}, ..., Z_{M_k}),$$

i.e. (C2) holds. By the convergence principle, we have that $\hat{P}_n \xrightarrow{D} Z$ in $(C[0,1], \|\cdot\|_\infty)$.

(d) (i) Note that $\mathbb{E}\hat{E}_n(t) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}[\sin(X_i t)] = \mathbb{E}[\sin(X_1 t)] = E(t)$ since X_i are iid, thus $\mathbb{E}\hat{P}_n(t) = 0$. We conclude (note that $\mathrm{Var}(Z) = \mathrm{Var}(Z + c)$ for constants c) that

$$\begin{aligned}
\mathbb{E}\big[|\hat{P}_n(t) - \hat{P}_n(s)|^2\big] &= \mathrm{Var}(\hat{P}_n(t) - \hat{P}_n(s)) = \mathrm{Var}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^n \big(\sin(X_i t) - \sin(X_i s)\big)\Big) \\
&\overset{X_i \text{ iid}}{=} \frac{1}{n}\sum_{i=1}^n \mathrm{Var}(\sin(X_i t) - \sin(X_i s)) = \mathrm{Var}(\sin(X_1 t) - \sin(X_1 s)) \\
&\overset{\mathrm{Var}(Z)=\mathbb{E}[Z^2]-\mathbb{E}[Z]^2}{\leq} \mathbb{E}[|\sin(X_1 t) - \sin(X_1 s)|^2] \leq \mathbb{E}[X_1^2] \cdot |s-t|^2,
\end{aligned}$$

where we used $|\sin(x) - \sin(y)| \leq |x - y|$.
(ii) By (i), we have for all $\varepsilon > 0$ that for all $s, t \in [0,1]$:

$$\mathbb{P}(|\hat{P}_n(s) - \hat{P}_n(t)| \geq \varepsilon) \leq \frac{\mathbb{E}[|\hat{P}_n(s) - \hat{P}_n(t)|^2]}{\varepsilon^2} \leq \frac{\mathbb{E}[X_1^2]}{\varepsilon^2}|s-t|^2,$$

i.e. (C1) is fulfilled with $\gamma = 1$.
Now let $k \in \mathbb{N}$, $0 \leq t_1 < ... < t_k \leq 1$. By the multivariate central limit theorem, we have

$$\begin{pmatrix} \hat{P}_n(t_1) \\ \vdots \\ \hat{P}_n(t_k) \end{pmatrix} = \frac{1}{\sqrt{n}}\sum_{i=1}^n \begin{pmatrix} \sin(X_i t_1) - \mathbb{E}[\sin(X_i t_1)] \\ \vdots \\ \sin(X_i t_k) - \mathbb{E}[\sin(X_i t_k)] \end{pmatrix} \xrightarrow{D} N(0,\ \Sigma),$$

where $\Sigma_{ij} = \mathrm{Cov}(\sin(X_1 t_1), \sin(X_1 t_k))$. So if we ask Z to have the covariance function $\gamma(s,t) = \mathbb{E}[Z_s Z_t] = \mathrm{Cov}(\sin(X_1 s), \sin(X_1 t))$, then $(Z_{t_1}, ..., Z_{t_k}) \sim N(0, \Sigma)$ (multivariate normal with mean 0 since Z is a centered Gaussian process). Thus with this choice of $\gamma(s,t)$ we have shown that

$$(\hat{P}_n(t_1), ..., \hat{P}_n(t_k)) \xrightarrow{D} (Z_{t_1}, ..., Z_{t_k}),$$

i.e. (C2) holds. By the convergence principle, we have that $\hat{P}_n \xrightarrow{D} Z$ in $(C[0,1], \|\cdot\|_\infty)$.
(iii) By the hint, $\Phi : (C[0,1], \|\cdot\|_\infty) \to (\mathbb{R}, |\cdot|), \Phi(f) := \sup_{t\in[0,1]} |f(t)|$ is continuous. By the continuous mapping theorem applied to (ii) we obtain that

$$\sqrt{n} \sup_{t\in[0,1]} |\hat{E}_n(t) - E(t)| = \sup_{t\in[0,1]} |\hat{P}_n(t)| = \Phi(\hat{P}_n) \xrightarrow{D} \Phi(Z) = \sup_{t\in[0,1]} |Z_t|.$$

For each $\omega \in \Omega$, the right hand side $\sup_{t\in[0,1]} |Z_t(\omega)|$ is finite since $t \mapsto Z_t(\omega)$ is continuous. By Slutzky's theorem, we obtain

$$\sup_{t\in[0,1]} |\hat{E}_n(t) - E(t)| = \frac{1}{\sqrt{n}} \cdot \sqrt{n} \sup_{t\in[0,1]} |\hat{E}_n(t) - E(t)| \xrightarrow{D} 0 \cdot \sup_{t\in[0,1]} |Z_t| = 0,$$

i.e. $\sup_{t\in[0,1]} |\hat{E}_n(t) - E(t)| \xrightarrow{\mathbb{P}} 0$ (the limit is constant). We obtain that

$$\big|\sup_{t\in[0,1]} \hat{E}_n(t) - \sup_{t\in[0,1]} E(t)\big| \leq \sup_{t\in[0,1]} |\hat{E}_n(t) - E(t)| \xrightarrow{\mathbb{P}} 0,$$

which implies that $\sup_{t\in[0,1]} \hat{E}_n(t) \xrightarrow{\mathbb{P}} \sup_{t\in[0,1]} E(t)$. Since $X_1 \sim U[0,1]$, we have $E(t) = \mathbb{E}[\sin(X_1 t)] = \int_0^1 \sin(xt)\ \mathrm{d}x = \frac{1-\cos(t)}{t}$ which is maximized on $[0,1]$ in $t=1$ (by the hint that $E(t)$ is nondecreasing in $[0,1]$, i.e. $\sup_{t\in[0,1]} E(t) = 1-\cos(1)$. This shows that

$$\sup_{t\in[0,1]} \hat{E}_n(t) \xrightarrow{\mathbb{P}} 1-\cos(1).$$

(e) (i) Note that $\mathbb{E}\hat{E}_n(t) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}[X_1^{1+t}] = \mathbb{E}[X_1^{1+t}] = E(t)$ since X_i are iid, thus $\mathbb{E}\hat{P}_n(t) = 0$. We conclude (note that $\mathrm{Var}(Z) = \mathrm{Var}(Z+c)$ for constants c) that

$$\begin{aligned}
\mathbb{E}\big[|\hat{P}_n(t) - \hat{P}_n(s)|^2\big] &= \mathrm{Var}(\hat{P}_n(t) - \hat{P}_n(s)) = \mathrm{Var}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^n \big(X_i^{1+t} - X_i^{1+s}\big)\Big) \\
&\overset{X_i \text{ iid}}{=} \frac{1}{n}\sum_{i=1}^n \mathrm{Var}(X_i^{1+t} - X_i^{1+s}) = \mathrm{Var}(X_1^{1+t} - X_1^{1+s}) \\
&\overset{\mathrm{Var}(Z)=\mathbb{E}[Z^2]-\mathbb{E}[Z]^2}{\leq} \mathbb{E}[|X_1^{1+t} - X_1^{1+s}|^2].
\end{aligned}$$

By a Taylor's expansion of $f(t) = x^{1+t}$, we have $f'(t) = \log(x)x^{1+t}$ and $f(t)-f(s) = (t-s)f'(\tilde{t})$ with $\tilde{t}$ between s,t. Thus we have (note that for $C>0$ large enough, we have $\log(x)^2x^4 \leq Cx^5$ for $x \geq 1$)

$$\mathbb{E}\big[|\hat{P}_n(t) - \hat{P}_n(s)|^2\big] \leq |t-s|^2\mathbb{E}[\log(X_1)^2|X_1|^{2(1+\tilde{t})}] \overset{\tilde{t}\in[0,1]}{\leq} |t-s|^2\mathbb{E}[\log(X_1)^2X_1^4] \leq C\mathbb{E}[X_1^5]\cdot|s-t|^2.$$

(ii) By (i), we have for all $\varepsilon > 0$ that for all $s,t \in [0,1]$:

$$\mathbb{P}(|\hat{P}_n(s) - \hat{P}_n(t)| \geq \varepsilon) \leq \frac{\mathbb{E}[|\hat{P}_n(s) - \hat{P}_n(t)|^2]}{\varepsilon^2} \leq \frac{C\mathbb{E}[X_1^5]}{\varepsilon^2}|s-t|^2,$$

i.e. (C1) is fulfilled with $\gamma = 1$.

Now let $k \in \mathbb{N}$, $0 \leq t_1 < ... < t_k \leq 1$. By the multivariate central limit theorem, we have

$$\begin{pmatrix} \hat{P}_n(t_1) \\ \vdots \\ \hat{P}_n(t_k) \end{pmatrix} = \frac{1}{\sqrt{n}}\sum_{i=1}^n \begin{pmatrix} X_i^{1+t_1} - \mathbb{E}[X_i^{1+t_1}] \\ \vdots \\ X_i^{1+t_k} - \mathbb{E}[X_i^{1+t_k}] \end{pmatrix} \xrightarrow{D} N(0,\ \Sigma),$$

where $\Sigma_{ij} = \mathrm{Cov}(X_1^{1+t_i}, X_1^{1+t_j}) = \mathbb{E}[X_1^{2+t_i+t_j}] - \mathbb{E}[X_1^{1+t_i}]\mathbb{E}[X_1^{1+t_j}] = E(t_i+t_j+1) - E(t_i)E(t_j)$. Since Z has covariance function $\gamma(s,t) = \mathbb{E}[Z_sZ_t] = E(s+t+1) - E(s)E(t)$, then $(Z_{t_1}, ..., Z_{t_k}) \sim N(0,\Sigma)$ (multivariate normal with mean 0 since Z is a centered Gaussian process) with the same Σ as above. Thus we have shown that

$$(\hat{P}_n(t_1), ..., \hat{P}_n(t_k)) \xrightarrow{D} (Z_{t_1}, ..., Z_{t_k}),$$

i.e. (C2) holds. By the convergence principle, we have that $\hat{P}_n \xrightarrow{D} Z$ in $(C[0,1], \|\cdot\|_\infty)$.

(f) (i) By Donsker's theorem, we have

$$P_n := (P_n(t))_{t\in[0,1]} := \Big(\frac{1}{\sqrt{n}}S_{\lfloor nt\rfloor} + \frac{1}{\sqrt{n}}(nt - \lfloor nt\rfloor)\varepsilon_{\lfloor nt\rfloor+1}\Big)_{t\in[0,1]} \xrightarrow{D} B$$

in $(C[0,1], \|\cdot\|_\infty)$. The mapping $\Phi : (C[0,1], \|\cdot\|_\infty) \to (\mathbb{R}, |\cdot|)$, $\Phi(f) := \int_0^1 |f(t)|\ \mathrm{d}t$ is Lipschitz continuous with constant 1 since

$$\begin{aligned}
\big|\Phi(f) - \Phi(g)\big| &\leq \int_0^1 \big||f(t)| - |g(t)|\big|\ \mathrm{d}t \leq \int_0^1 |f(t) - g(t)|\ \mathrm{d}t \quad (*) \\
&\leq \|f-g\|_\infty.
\end{aligned}$$

By the continuous mapping theorem, we obtain

$$\Phi(P_n) \xrightarrow{D} \Phi(B) = \int_0^1 |B_t| \, \mathrm{d}t.$$

Furthermore, we have by (*),

$$\begin{aligned}\Big|\Phi(P_n) - \Phi\big((\frac{1}{\sqrt{n}}S_{\lfloor nt\rfloor})_{t\in[0,1]}\big)\Big| &\leq \frac{1}{\sqrt{n}}\int_0^1 \big|(nt-\lfloor nt\rfloor)\cdot\varepsilon_{\lfloor nt\rfloor+1}\big| \, \mathrm{d}t \\ &\leq \frac{1}{\sqrt{n}}\Big|\sum_{k=1}^n |\varepsilon_k| \int_{(k-1)/n}^{k/n} (nt-(k-1)) \, \mathrm{d}t\Big| \\ &= \frac{1}{2\sqrt{n}} \underbrace{\frac{1}{n}\sum_{k=1}^n |\varepsilon_k|}_{\xrightarrow{\mathbb{P}}\mathbb{E}|\varepsilon_1|} \xrightarrow{\mathbb{P}} 0\end{aligned}$$

by the weak law of large numbers. By Slutzky's theorem, we have

$$\begin{aligned}\frac{1}{n^{3/2}}\sum_{k=1}^{n-1}|S_k| &= \frac{1}{\sqrt{n}}\sum_{k=1}^n \int_{(k-1)/n}^{k/n} |S_{k-1}| \, \mathrm{d}t = \frac{1}{\sqrt{n}}\int_0^1 |S_{\lfloor nt\rfloor}| \, \mathrm{d}t \\ &= \Phi\big((\frac{1}{\sqrt{n}}S_{\lfloor nt\rfloor})_{t\in[0,1]}\big) \xrightarrow{D} \Phi(B) = \int_0^1 |B_t| \, \mathrm{d}t\end{aligned}$$

(ii) By Donsker's theorem, we have

$$P_n := (P_n(t))_{t\in[0,1]} := \Big(\frac{1}{\sqrt{n}}S_{\lfloor nt\rfloor} + \frac{1}{\sqrt{n}}(nt-\lfloor nt\rfloor)\varepsilon_{\lfloor nt\rfloor+1}\Big)_{t\in[0,1]} \xrightarrow{D} B$$

The mapping $\Phi : (C[0,1], \|\cdot\|_\infty) \to (\mathbb{R}, |\cdot|)$, $\Phi(f) := \inf_{t\in[0,1]} f(t)$ is continuous: Note that

$$\begin{aligned}&\inf_{t\in[0,1]} f(t) \geq \inf_{t\in[0,1]}\big\{f(t)-g(t)\big\} + \inf_{t\in[0,1]} g(t) \\ \Rightarrow\ &\inf_{t\in[0,1]} g(t) - \inf_{t\in[0,1]} f(t) \leq -\inf_{t\in[0,1]}\big\{f(t)-g(t)\big\} \leq \sup_{t\in[0,1]}|f(t)-g(t)|\end{aligned}$$

Swapping the roles of f, g we obtain the same inequality but with different sign on the left hand side, leading to

$$\big|\Phi(f) - \Phi(g)\big| \leq \|f-g\|_\infty.$$

This shows that Φ is even Lipschitz continuous with Lipschitz constant 1.
Application of the continuous mapping theorem yields

$$\Phi(P_n) \xrightarrow{D} \Phi(B) = \inf_{t\in[0,1]} |B_t|.$$

Furthermore, we have by the Lipschitz continuity of Φ,

$$\begin{aligned}\Big|\Phi(P_n) - \Phi\Big((\frac{1}{\sqrt{n}}S_{\lfloor nt\rfloor})_{t\in[0,1]}\Big)\Big| &\leq \frac{1}{\sqrt{n}}\sup_{t\in[0,1]}\big|nt-\lfloor nt\rfloor\big|\cdot|\varepsilon_{\lfloor nt\rfloor+1}| \\ &\leq \frac{1}{\sqrt{n}}\max_{k=1,\ldots,n}|\varepsilon_k| \xrightarrow{\mathbb{P}} 0\end{aligned}$$

by the hint. By Slutzky's theorem, we have

$$\frac{1}{\sqrt{n}}\min_{k=0,\ldots,n} S_k = \inf_{t\in[0,1]}\frac{1}{\sqrt{n}}S_{\lfloor nt\rfloor} = \Phi\big((\frac{1}{\sqrt{n}}S_{\lfloor nt\rfloor})_{t\in[0,1]}\big) \xrightarrow{D} \inf_{t\in[0,1]} B_t$$

which gives the result.
(iii) By Donsker's theorem, we have

$$P_n := (P_n(t))_{t\in[0,1]} := \Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}(\varepsilon_i-\mu) + \frac{1}{\sqrt{n}}(nt-\lfloor nt\rfloor)(\varepsilon_{\lfloor nt\rfloor+1}-\mu)\Big)_{t\in[0,1]} \xrightarrow{D} B$$

in $(C[0,1],\|\cdot\|_\infty)$ since $\mathbb{E}[(\varepsilon_1-\mu)^4]<\infty$ and $\mathbb{E}(\varepsilon_1-\mu)=0$. The mapping $\Phi : (C[0,1],\|\cdot\|_\infty) \to (C[0,1],\|\cdot\|_\infty)$, $\Phi(f) := \{t\mapsto f(t)-tf(1)\}$ is Lipschitz continuous with constant 2 since

$$\big|\Phi(f)-\Phi(g)\big| \le \sup_{t\in[0,1]}|f(t)-g(t)| + \sup_{t\in[0,1]}|t|\cdot|f(1)-g(1)| \le 2\|f-g\|_\infty.$$

By the continuous mapping theorem, we obtain (from the lecture it is known that $(B_t-tB_1)_{t\in[0,1]}$ is a Brownian Bridge):

$$\Phi(P_n) \xrightarrow{D} \Phi(B) = (B_t-tB_1)_{t\in[0,1]} \stackrel{d}{=} B^\circ.$$

Here, we have (the μ's cancel out!)

$$\begin{aligned}
\Phi(P_n)_t &= \Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}(\varepsilon_i-\mu) + \frac{1}{\sqrt{n}}(nt-\lfloor nt\rfloor)(\varepsilon_{\lfloor nt\rfloor+1}-\mu)\Big) - t\cdot\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\varepsilon_i-\mu)\Big)\\
&= \Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\varepsilon_i + \frac{1}{\sqrt{n}}(nt-\lfloor nt\rfloor)\varepsilon_{\lfloor nt\rfloor+1}\Big) - t\cdot\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i\Big)\\
&= \frac{1}{\sqrt{n}}(S_{\lfloor nt\rfloor}-t\cdot S_n) + \frac{1}{\sqrt{n}}(nt-\lfloor nt\rfloor)\varepsilon_{\lfloor nt\rfloor+1} = R_n(t),
\end{aligned}$$

which gives the desired convergence result.