Exam preparation sheet - Solutions part 2

0.2 Brownian motion and its properties

Solutions: (a) (i) We have to show that X satisfies the conditions of a Brownian motion:

- Since $t \mapsto B$ is continuous, the same holds for the continuous composition $t \mapsto B_{1-t} - B_1 = X_t$,
- $X_0 = B_{1-0} - B_1 = 0$,
- For $t \in [0,1]$, we have $\mathbb{E}X_t = \mathbb{E}B_{1-t} - \mathbb{E}B_1 = 0$ and for $s,t \in [0,1]$:

$$\begin{aligned}\operatorname{Cov}(X_s, X_t) = \operatorname{Cov}(B_{1-s} - B_1, B_{1-t} - B_1) &= \operatorname{Cov}(B_{1-s}, B_{1-t}) - \operatorname{Cov}(B_{1-s}, B_1) \\ &\quad -\operatorname{Cov}(B_1, B_{1-t}) + \operatorname{Cov}(B_1, B_1) \\ &= (1-s) \wedge (1-t) - (1-s) - (1-t) + 1 \\ &= s + t - (s \vee t) \\ &= s \wedge t,\end{aligned}$$

  i.e. X has the same mean and covariance functions as a Brownian motion.

- Fix $0 \leq t_1 < t_2 < ... < t_n$. Since B is a Brownian motion, we have that $(B_1, B_{1-t_1}, ..., B_{1-t_n})' \sim N(0, \Sigma)$ with some matrix Σ (we have already seen that the expectation is 0). Since

$$\begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -1 & 0 & \dots & 0 & 1 \end{pmatrix}}_{=:A} \cdot \begin{pmatrix} B_1 \\ B_{1-t_1} \\ \vdots \\ B_{1-t_n} \end{pmatrix} \sim N(0, A\Sigma A'),$$

  thus $(X_{t_1}, ..., X_{t_n})'$ is multivariate Gaussian distributed.

(ii)

- Since $t \mapsto B$ is continuous, $t \mapsto B_{t_0} + Z(B_t - B_{t_0})$ is continuous as a composition of continuous functions. This shows that $t \mapsto X_t$ is continuous in every point $t \in [0,1] \backslash \{t_0\}$. Since $B_{t_0} = B_{t_0} + Z(B_{t_0} - B_{t_0})$, the two cases in the definition of X_t coincide for $t = t_0$, thus $t \mapsto X_t$ is continuous.
- Since $0 < t_0$, $X_0 = B_0 = 0$,
- (It is not necessary to prove the following. Everything also follows from the next point) For $t \in [0,1]$, we have

$$\mathbb{E}X_t = \begin{cases} \mathbb{E}B_t = 0, & t < t_0 \\ \mathbb{E}B_{t_0} + \mathbb{E}[Z(B_t - B_{t_0})] = \mathbb{E}B_{t_0} + \mathbb{E}Z\mathbb{E}(B_t - B_{t_0}) = 0, & t \geq t_0, \end{cases}$$

  since Z, B are independent and $\mathbb{E}B_{t_0} = 0 = \mathbb{E}(B_t - B_{t_0})$.
  For $s,t \in [0,1]$ we have three cases: For $s,t < t_0$, we have $\operatorname{Cov}(X_t, X_s) = \operatorname{Cov}(B_t, B_s) = \min\{s,t\}$ since B is a Brownian motion.
  For $s < t_0 \leq t$, we have

$$\begin{aligned}\operatorname{Cov}(X_t, X_s) &= \operatorname{Cov}\big(B_s, B_{t_0} + Z(B_t - B_{t_0})\big) = \operatorname{Cov}(B_s, B_{t_0}) + \operatorname{Cov}(B_s, Z(B_t - B_{t_0})) \\ &= \min\{s, t_0\} + \underbrace{\mathbb{E}[ZB_s(B_t - B_{t_0})]}_{=\mathbb{E}[Z]\cdot\mathbb{E}[B_S(B_t - B_{t_0})=0} = s = \min\{s,t\}.\end{aligned}$$

In the case $t_0 \le s,t$, we have

$$\begin{aligned}
\mathrm{Cov}(X_s, X_t) &= \mathrm{Cov}\big(B_{t_0} + Z(B_s - B_{t_0}), B_{t_0} + Z(B_t - B_{t_0})\big) \\
&= \mathrm{Cov}(B_{t_0}, B_{t_0}) + \underbrace{\mathrm{Cov}(B_{t_0}, Z(B_t - B_{t_0})) + \mathrm{Cov}(B_{t_0}, Z(B_s - B_{t_0}))}_{=0+0 \text{ (like above)}} \\
&\quad + \mathrm{Cov}(Z(B_s - B_{t_0}), Z(B_t - B_{t_0})) \\
&= t_0 + \mathbb{E}[Z^2 (B_s - B_{t_0})(B_t - B_{t_0})] \\
&= t_0 + \underbrace{\mathbb{E}[Z^2]}_{=1}\big(\mathbb{E}[B_s B_t] + \mathbb{E}[B_{t_0} B_{t_0}] - \mathbb{E}[B_s B_{t_0}] - \mathbb{E}[B_t B_{t_0}]\big) \\
&= t_0 + \big(\min\{s,t\} + \min\{t_0, t_0\} - \min\{s, t_0\} - \min\{t, t_0\}\big) \\
&= t_0 + \min\{s,t\} + t_0 - t_0 - t_0 = \min\{s,t\}.
\end{aligned}$$

i.e. X has the same mean and covariance functions as a Brownian motion.

- Fix $n \in \mathbb{N}$. Let $k \in \{1, ..., n\}$ be such that $0 \le t_1 < t_2 < ... < t_k < t_0 \le t_{k+1} < ... < t_n \le 1$. Let $A_1, ..., A_n \in \mathcal{B}(\mathbb{R})$. Then we have, since $Z \in \{-1, 1\}$ is independent of B,

$$\begin{aligned}
&\mathbb{P}(X_{t_1} \in A_1, ..., X_{t_n} \in A_n) \\
=\ &\mathbb{P}\big(\forall i \in \{1,...,k\} : B_{t_i} \in A_i, \quad \forall i \in \{k+1, ..., n\} : B_{t_0} + Z(B_{t_i} - B_{t_0}) \in A_i\big) \\
=\ &\mathbb{P}\big(\forall i \in \{1,...,k\} : B_{t_i} \in A_i, \quad \forall i \in \{k+1, ..., n\} : B_{t_0} + Z(B_{t_i} - B_{t_0}) \in A_i, Z = 1\big) \\
&+ \mathbb{P}\big(\forall i \in \{1,...,k\} : B_{t_i} \in A_i, \quad \forall i \in \{k+1, ..., n\} : B_{t_0} + Z(B_{t_i} - B_{t_0}) \in A_i, Z = -1\big) \\
=\ &\mathbb{P}\big(\forall i \in \{1,...,n\} : B_{t_i} \in A_i\big)\mathbb{P}(Z = 1) \\
&+ \mathbb{P}\big(\forall i \in \{1,...,k\} : B_{t_i} \in A_i, \quad \forall i \in \{k+1, ..., n\} : 2B_{t_0} - B_{t_i} \in A_i\big)\mathbb{P}(Z = -1) \quad (*).
\end{aligned}$$

Note that $(B_{t_1}, ..., B_{t_n}) \sim N(0, \Sigma)$ with $\Sigma_{ij} = \min\{t_i, t_j\}$ since B is a centered Gaussian process with covariance function $\mathbb{E} B_s B_t = \min\{s,t\}$. We also have

$$\begin{pmatrix} B_{t_1} \\ \vdots \\ B_{t_k} \\ 2B_{t_0} - B_{t_{k+1}} \\ \vdots \\ 2B_{t_0} - B_{t_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \vdots & 0 & \vdots & & & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \vdots & & & \vdots \\ 0 & \dots & 0 & 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & 2 & -1 & 0 & \dots & 0 \\ \vdots & & & \vdots & 2 & 0 & \ddots & \ddots & \vdots \\ \vdots & & & \vdots & 2 & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 2 & 0 & \dots & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} B_{t_1} \\ \vdots \\ B_{t_k} \\ B_{t_0} \\ B_{t_{k+1}} \\ \vdots \\ B_{t_n} \end{pmatrix} \sim N(0, \Sigma)$$

with the same Σ as above since for $i \le k$, $j \ge k+1$ we have

$$\mathrm{Cov}(B_{t_i}, 2B_{t_0} - B_{t_j}) = 2\mathrm{Cov}(B_{t_i}, B_{t_0}) - \mathrm{Cov}(B_{t_i}, B_{t_j}) = 2t_i - t_i = t_i = \min\{t_i, t_j\},$$

and for $i, j \ge k+1$:

$$\begin{aligned}
&\mathrm{Cov}(2B_{t_0} - B_{t_i}, 2B_{t_0} - B_{t_j}) \\
=\ &4\mathrm{Cov}(B_{t_0}, B_{t_0}) - 2\mathrm{Cov}(B_{t_0}, B_{t_i}) - 2\mathrm{Cov}(B_{t_0}, B_{t_j}) + \mathrm{Cov}(B_{t_i}, B_{t_j}) \\
=\ &4t_0 - 2t_0 - 2t_0 + \min\{t_i, t_j\} = \min\{t_i, t_j\}.
\end{aligned}$$

This shows that

$$
\begin{aligned}
&\mathbb{P}(X_{t_1} \in A_1, ..., X_{t_n} \in A_n) \\
\overset{(*)}{=}\ & \frac{1}{2}\Big(\mathbb{P}\big(\forall i \in \{1, ..., n\} : B_{t_i} \in A_i\big) \\
&\quad +\mathbb{P}\big(\forall i \in \{1, ..., k\} : B_{t_i} \in A_i, \quad \forall i \in \{k+1, ..., n\} : 2B_{t_0} - B_{t_i} \in A_i\big)\Big) \\
=\ & \frac{1}{2}\Big[\mathbb{P}\Big(N(0, \Sigma) \in A_1 \times ... \times A_n\Big) + \mathbb{P}\Big(N(0, \Sigma) \in A_1 \ \times ... \times A_n\Big)\Big] \\
=\ & \mathbb{P}(N(0, \Sigma) \in A_1 \times ... \times A_n)
\end{aligned}
$$

Since $\{A_1 \times ... \times A_n : A_i \in \mathcal{B}(\mathbb{R})$ for $i = 1, ..., n\}$ is a generating system of $\mathcal{B}(\mathbb{R}^n)$, we have shown that $(X_{t_1}, ..., X_{t_n}) \sim N(0, \Sigma)$, i.e. $(X_{t_1}, ..., X_{t_n})$ has the same distribution as $(B_{t_1}, ..., B_{t_n})$.

(iii) We have to show that $(X_t)_{t\geq 0}$ satisfies the properties of a standard Brownian motion. Here, we use the original definition with independent increments.

(1) Since $(B_t), (B'_t)$ are Brownian motions, it holds that $B_0 = B'_0 = 0$. Thus $X_0 = a \cdot B_0 + b \cdot B'_0 = 0 + 0 = 0$.

(2) Since $(B_t), (B'_t)$ are Brownian motions, $t \mapsto B_t$ and $t \mapsto B'_t$ are continuous. Since '+' is a continuous operation, $t \mapsto X_t = B_t + B'_t$ is continuous as well.

(3) Let $n \in \mathbb{N}$, $0 \leq t_1 < ... < t_n$. Since $(B_t), (B'_t)$ are Brownian motions, $(B_{t_i} - B_{t_{i-1}})_{i=1,...,n}$ are independent and $(B'_{t_i} - B'_{t_{i-1}})_{i=1,...,n}$ are independent. Since $(B_t), (B'_t)$ are independent, we conclude that the $2n$ random variables are jointly independent:

$$
\begin{aligned}
\mathbb{P}^{(B_{t_i}-B_{t_{i-1}})_{i=1,...,n},(B'_{t_i}-B'_{t_{i-1}})_{i=1,...,n}} &= \mathbb{P}^{(B_{t_i}-B_{t_{i-1}})_{i=1,...,n}} \otimes \mathbb{P}^{(B'_{t_i}-B'_{t_{i-1}})_{i=1,...,n}} \\
&= \left(\bigotimes_{i=1}^{n} \mathbb{P}^{B_{t_i}-B_{t_{i-1}}}\right) \otimes \left(\bigotimes_{i=1}^{n} \mathbb{P}^{B'_{t_i}-B'_{t_{i-1}}}\right).
\end{aligned}
$$

Thus $(X_{t_i} - X_{t_{i-1}})_{i=1,...,n} = ((B'_{t_i} - B'_{t_{t-1}}) + (B_{t_i} - B_{t_{t-1}}))_{i=1,...,n}$ are independent as combinations of different independent random variables.

(4) Let $s \leq t$. Since $(B_t), (B'_t)$ are Brownian motions, we have $B_t - B_s \sim N(0, t-s)$ and $B'_t - B'_s \sim N(0, t-s)$. Since $(B_t - B_s), (B'_t - B'_s)$ are independent, we conclude that

$$
X_t - X_s = a \cdot (B_t - B_s) + b \cdot (B'_t - B'_s) \sim N(0, a^2(t-s) + b^2(t-s)) \overset{a^2+b^2=1}{=} N(0, t-s).
$$

(iv) Since $r(\cdot)$ is continuous and strictly increasing with $r(0) = 0$, we have that the inverse $r^{-1}(\cdot)$ is continuous and strictly increasing with $r^{-1}(0)$. We now show that X satisfies the characterizing conditions of a Brownian motion:

- $X_0 = \frac{W_{r^{-1}(0)}}{v(r^{-1}(0))} = \frac{W_0}{v(0)} = 0$ since $W_0 = 0$.
- Since $v \in C[0, \infty)$ and r^{-1} is continuous and W is continuous, we have that $t \mapsto \frac{W_{r^{-1}(t)}}{v(r^{-1}(t))} = X_t$ is continuous as a composition of continuous functions.
- Since W is a centered Gaussian process with covariance function $\gamma(s, t) = u(s)v(t)$, it holds that $\mathbb{E}W_t = 0$ and $\text{Cov}(W_s, W_t) = u(s)v(t)$ for all $0 \leq s \leq t$. We conclude that for all $t \geq 0$,

$$
\mathbb{E}[X_t] = \frac{\mathbb{E}W_{r^{-1}(t)}}{v(r^{-1}(t))} = 0,
$$

and for all $0 \le s \le t$ (note that $r^{-1}(s) \le r^{-1}(t)$ since r^{-1} is nondecreasing)

$$\begin{aligned}
\mathrm{Cov}(X_t, X_s) &= \frac{1}{v(r^{-1}(t))v(r^{-1}(s))}\mathrm{Cov}(W_{r^{-1}(t)}, W_{r^{-1}(s)}) \\
&= \frac{1}{v(r^{-1}(t))v(r^{-1}(s))}u(r^{-1}(s))v(r^{-1}(t)) \\
&= \frac{u(r^{-1}(s))}{v(r^{-1}(s))} = r(r^{-1}(s)) = s = \min\{s,t\}
\end{aligned}$$

which shows that X has the covariance function of a Brownian motion.

- Let $n \in \mathbb{N}$, $0 \le t_0 < t_1 < ... < t_n \le 1$. Since W is a centered Gaussian process, we have that $(W_{r^{-1}(t_1)}, ..., W_{r^{-1}(t_n)})' \sim N(0,\Sigma)$ with some $\Sigma \in \mathbb{R}^{n\times n}$. We conclude that

$$\begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{v(r^{-1}(t_1))} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{v(r^{-1}(t_n))} \end{pmatrix}}_{=:A} \cdot \begin{pmatrix} W_{r^{-1}(t_1)} \\ \vdots \\ W_{r^{-1}(t_n)} \end{pmatrix} \sim N(0, A\Sigma A')$$

which shows that X is a (centered) Gaussian process.

(b) We have to show that X satisfies the characterizing conditions of a Brownian Bridge:

- Since $t \mapsto B$ is continuous, the same holds for the continuous composition $t \mapsto (1-t)B_{\frac{t}{1-t}}$ for $t \in [0,1)$. For $t = 1$, we have

$$\lim_{t\to 1} X_t = \lim_{t\to 1} \underbrace{t}_{\to 1} \cdot \left(\frac{1-t}{t}\right) \cdot B_{\frac{1}{(\frac{1-t}{t})}} \overset{s:=\frac{1-t}{t}}{=} \lim_{s\to 0} s \cdot B_{1/s} = 0$$

since $W_s := sB_{1/s}$ is a Brownian motion (time reverse) and thus continuous in 0.

- $X_0 = (1-0)\cdot B_0 = 0$,

- For $t \in [0,1)$, we have $\mathbb{E}X_t = (1-t)\mathbb{E}B_{\frac{t}{1-t}} = 0$ (trivially $\mathbb{E}X_1 = 0$). Note that $t \mapsto \frac{t}{1-t}$ is increasing for $t \in [0,1)$. Thus we have for $s \le t \in [0,1)$:

$$\begin{aligned}
\mathrm{Cov}(X_s, X_t) &= \mathrm{Cov}((1-t)B_{\frac{t}{1-t}}, (1-s)B_{\frac{s}{1-s}}) = (1-t)(1-s)\cdot\frac{s}{1-s} \\
&= s(1-t) = \min\{s,t\} - st
\end{aligned}$$

i.e. X has the same mean and covariance functions as a Brownian motion (the case $s = 1$ or $t = 1$ is trivial since $X_1 = 0$ and thus the covariance is 0).

- Fix $0 \le t_1 < t_2 < ... < t_n$. Since B is a Brownian motion, we have that $(X_{t_1}, ..., X_{t_n})' = (B_{\frac{t_1}{1-t_1}}, ..., B_{\frac{t_n}{1-t_n}})' \sim N(0,\Sigma)$ with some matrix Σ (we have already seen that the expectation is 0), thus $(X_{t_1}, ..., X_{t_n})'$ is multivariate Gaussian distributed.

(c) (i) We have to show the three properties of a martingale: We have for all $t \ge 0$:

- Since $B_t \in \mathcal{F}_t$, we have that $X_t = B_t^2 - t \in \mathcal{F}_t$ as a composition of measurable functions.

- $\mathbb{E}|X_t| \le \mathbb{E}[B_t^2] + t = t + t = 2t < \infty$ (since $B_t \sim N(0,t)$).

- We have that $B_t - B_s \sim N(0, t-s)$ is independent of $\mathcal{F}_s$, and B_s is $\mathcal{F}_s$-measurable. Thus

$$\begin{aligned}
\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}[B_t^2 - t|\mathcal{F}_s] &= \mathbb{E}[(B_t - B_s)^2 + 2B_tB_s - B_s^2 - t|\mathcal{F}_s] \\
&= \mathbb{E}[(B_t - B_s)^2 + 2(B_t - B_s)B_s + B_s^2 - t|\mathcal{F}_s] \\
&= \underbrace{\mathbb{E}[(B_t - B_s)^2]}_{=t-s} + 2B_s \underbrace{\mathbb{E}[B_t - B_s]}_{=0} + B_s^2 - t \\
&= B_s^2 - s = X_s.
\end{aligned}$$

(ii) We have to show the three properties of a martingale: We have for all $t \geq 0$:

- By continuity, we have $\int_0^t B_s \, \mathrm{d}s = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n B_{t\cdot\frac{k}{n}}$. Since $t\frac{k}{n} \leq t$, $B_{t\cdot\frac{k}{n}} \in \mathcal{F}_t$. Thus $\frac{1}{n}\sum_{k=1}^n B_{t\cdot\frac{k}{n}} \in \mathcal{F}_t$ for all $n \in \mathbb{N}$ which implies that $\int_0^t B_s \, \mathrm{d}s \in \mathcal{F}_t$. We conclude that $X_t = tB_t - \int_0^t B_s \, \mathrm{d}s \in \mathcal{F}_t$ as a measurable composition of $\mathcal{F}_t$-measurable functions.

- With Fubini's theorem, we have $\mathbb{E}|X_t| \leq t\mathbb{E}|B_t| + \int_0^t \mathbb{E}|B_s| \, \mathrm{d}s \leq t\mathbb{E}[B_t^2]^{1/2} + \int_0^t \mathbb{E}[B_s^2]^{1/2} \, \mathrm{d}s \leq t^{3/2} + \int_0^t s^{1/2} \, \mathrm{d}s < \infty$ (since $B_s \sim N(0,s)$ for $0 \leq s \leq t$).

- We have

$$\mathbb{E}[X_t|\mathcal{F}_s] = t\mathbb{E}[B_t|\mathcal{F}_s] - \mathbb{E}[\int_s^t B_u - B_s \, \mathrm{d}u + B_s(t-s) + \int_0^s B_u \, \mathrm{d}u|\mathcal{F}_s].$$

By the first point, we have that $\int_0^s B_u \, \mathrm{d}u \in \mathcal{F}_s$. By Fubini's theorem (it holds that

$$\mathbb{E}\int_s^t |B_u - B_s| \, \mathrm{d}u \leq \int_s^t \underbrace{\mathbb{E}|B_u - B_s|}_{\leq \mathbb{E}[(B_u - B_s)^2]^{1/2} = (u-s)^{1/2}} \, \mathrm{d}u \leq \int_s^t (u-s)^{1/2} \, \mathrm{d}u < \infty$$

) we have $\mathbb{E}[\int_s^t B_u - B_s \, \mathrm{d}u|\mathcal{F}_s] = \int_s^t \mathbb{E}[B_u - B_s|\mathcal{F}_s] \, \mathrm{d}u = \int_s^t \mathbb{E}[B_u - B_s] \, \mathrm{d}u = 0$. We conclude that

$$\mathbb{E}[X_t|\mathcal{F}_s] = t\big(\underbrace{\mathbb{E}[B_t - B_s|\mathcal{F}_s]}_{=\mathbb{E}[B_t - B_s]=0} + B_s\big) - B_s(t-s) - \int_0^s B_u \, \mathrm{d}u = sB_s - \int_0^s B_u \, \mathrm{d}u = X_s.$$

(iii) We have to show the three properties of a martingale: For all $t \geq 0$,

- $\mathbb{E}|X_t| = \exp(-\frac{\lambda^2}{2}t) \cdot \mathbb{E}\exp(\lambda B_t) \overset{\text{hint}}{=} \exp(-\frac{\lambda^2}{2}t) \cdot \exp(\frac{1}{2}t\lambda^2) = 1 < \infty$ (note that $\lambda B_t \sim N(0, t\lambda^2)$).

- $X_t \in \mathcal{F}_t$ since $B_t \in \mathcal{F}_t$ (so X_t is a composition of measurable functions).

- For $0 \leq s \leq t$, we know that $\exp(\sigma(B_t - B_s))$ is independent of $\mathcal{F}_s$ (see Exercise Sheet 4, Task 16). Therefore,

$$\begin{aligned}
\mathbb{E}[X_t|\mathcal{F}_s] &= S_0 \exp\Big(-\frac{\lambda^2}{2}t\Big) \cdot \mathbb{E}[\exp(\lambda(B_t - B_s)) \cdot \exp(\lambda B_s)|\ \mathcal{F}_s] \\
&= S_0 \exp\Big(-\frac{\lambda^2}{2}t\Big) \cdot \exp(\lambda B_s) \cdot \mathbb{E}[\exp(\lambda(B_t - B_s))] \\
&\overset{\text{hint}}{=} S_0 \exp\Big(-\frac{\lambda^2}{2}t\Big) \cdot \exp(\lambda B_s) \cdot \exp\Big(\frac{1}{2}\lambda^2(t-s)\Big) \\
&= S_0 \exp\Big(\lambda B_s - \frac{\lambda^2}{2}s\Big) = X_s.
\end{aligned}$$

(d) (i) First possibility (Elementary argumentation): We use the generating system $\mathcal{E} := \{(-\infty, x] : x \in \mathbb{R}\}$ of $\mathcal{B}(\mathbb{R})$. For arbitrary $(-\infty, x] \in \mathcal{E}$, we have

$$\begin{aligned}\Phi^{-1}(-\infty, x] &= \{f \in C[0,1] : \max_{0 \le s \le T} f(s) \le x\} = \{f \in C[0,1] : \forall s \in [0,T] : f(s) \le x\} \\ &\overset{f \text{ cont.}}{=} \{f \in C[0,1] : \forall s \in [0,T] \cap \mathbb{Q} : f(s) \le x\} \\ &= \bigcap_{s \in [0,T] \cap \mathbb{Q}} \underbrace{\{f \in C[0,1] : f(s) \le x\}}_{\pi_t^{-1}(-\infty, x]} \in \mathcal{B}(C[0,1])\end{aligned}$$

as a countable intersection of elements of $\mathcal{B}(C[0,1])$ (recall that $\mathcal{B}(C[0,1])$ is generated by the projections $\pi_t : C[0,1] \to \mathbb{R}, f \mapsto f(t)$).
Second possibility (continuity): Recall that $\mathcal{B}(C[0,1])$ is generated by the open sets in $(C[0,1], \|\cdot\|_\infty)$. Recall that $\mathcal{E} := \{U \subset \mathbb{R} : U \text{ open}\}$ is a generating system of $\mathcal{B}(\mathbb{R})$. If we show that Φ is continuous, then it follows that for each $U \in \mathcal{E}$, $\Phi^{-1}(U)$ is open in $(C[0,1], \|\cdot\|_\infty)$ and thus $\Phi^{-1}(U) \in \mathcal{B}(C[0,1])$ which shows measurability of Φ.
It remains to show that Φ is continuous. Note that

$$|\Phi(f) - \Phi(g)| = \big| \max_{0 \le s \le T} f(s) - \max_{0 \le s \le T} g(s) \big| \overset{(***)}{\le} \max_{0 \le s \le T} |f(s) - g(s)| = \|f - g\|_\infty,$$

i.e. Φ is Lipschitz continuous. [Note that $f = f - g + g$ implies

$$\begin{aligned}&\max_{0 \le s \le T} f(s) \le \max_{0 \le s \le T} (f(s) - g(s)) + \max_{0 \le s \le T} g(s) \\ \Rightarrow\ &\max_{0 \le s \le T} f(s) - \max_{0 \le s \le T} g(s) \le \max_{0 \le s \le T} (f(s) - g(s)) \le \max_{0 \le s \le T} |f(s) - g(s)|\end{aligned}$$

which leads to (***) if we swap the roles of f, g and use both obtained inequalities].
(ii) By the lecture it is known that $W = (W_t)_{t \ge 0}$ with $W_t := \sqrt{T} B_{t/T}$ is again a Brownian motion (scaling invariance). By (i) we conclude that

$$M_T = \Phi(B) \overset{d}{=} \Phi(W) = \max_{0 \le s \le T} W_s = \sqrt{T} \max_{0 \le s \le T} B_{s/T} = \sqrt{T} \max_{0 \le s \le 1} B_s = \sqrt{T} M_1.$$

(iii) By the lecture it is known that $W = (W_t)_{t \ge 0}$ with $W_t := -B_t$ is again a Brownian motion (symmetry w.r.t. the x-axis). By (i) we conclude that

$$M_T = \Phi(B) \overset{d}{=} \Phi(W) = \max_{0 \le s \le T} W_s = \max_{0 \le s \le T} (-B_s) = -\min_{0 \le s \le T} B_s = -m_T.$$

Thus $\mathbb{E}M_T = -\mathbb{E}m_T$, or equivalently $\mathbb{E}[M_T + m_T] = 0$.

(e) (i) We use the generating system $\mathcal{E} := \{(x, 1] : x \in [0,1]\} \cup \{\infty\}$ of $\mathcal{B}([0,1] \cup \{\infty\})$. For arbitrary $(x, 1] \in \mathcal{E}$, we have

$$\begin{aligned}\Phi^{-1}(x, 1] &= \{f \in C[0,1] : \inf\{t \in [0,1] : f(t) < -1\} > x\} \\ &= \{f \in C[0,1] : \forall s \in [0,x] : f(s) \ge -1\} \\ &\overset{f \text{ cont.}}{=} \{f \in C[0,1] : \forall s \in [0,x] \cap \mathbb{Q} : f(s) \ge -1\} \\ &= \bigcap_{s \in [0,x] \cap \mathbb{Q}} \underbrace{\{f \in C[0,1] : f(s) \ge -1\}}_{\pi_t^{-1}[-1,\infty) \in \mathcal{B}(C[0,1])} \in \mathcal{B}(C[0,1]).\end{aligned}$$

For $\{\infty\} \in \mathcal{E}$, we have $\Phi^{-1}\{\infty\} = \{f \in C[0,1] : \forall s \in [0,1] : f(s) \ge -1\} \in \mathcal{B}(C[0,1])$ as above.
(ii) By the lecture, it is known that $B \overset{d}{=} W := (-B_t)_{t \ge 0}$. We obtain that

$$\tau_1 = \Phi(B) \overset{d}{=} \Phi(W) = \inf\{t \ge 0 : -B_t < -1\} = \inf\{t \ge 0 : B_t > 1\} = \tau_2.$$

(f) (i) Note that if $f \in C[0,1]$, then $\big[\forall s,t \in [0,1]\cap\mathbb{Q} : |f(s)-f(t)| \le C|s-t|^\alpha\big]$ already implies the same statement for all $s,t \in [0,1]$. To prove this, let $s,t \in [0,1]$ be arbitrary and $(s_n),(t_n) \subset \mathbb{Q}$ sequences with $s_n \to s$, $t_n \to t$. Then we have $|f(s_n)-f(t_n)| \le C|s_n-t_n|^\alpha$ for all $n \in \mathbb{N}$. Taking $\lim_{n\to\infty}$ on both sides and using the continuity of f, we obtain $|f(s)-f(t)| \le C|s-t|^\alpha$. (*)
Now, note that

$$\begin{aligned} M &= \{\omega \in \Omega : \forall s,t : |B_s(\omega)-B_t(\omega)| \le C|s-t|^\alpha\} \\ &\overset{(*)}{=} \{\omega \in \Omega : \forall s,t \in \mathbb{Q}\cap[0,1] : |B_s(\omega)-B_t(\omega)| \le C|s-t|^\alpha\} \\ &= \bigcap_{s,t\in\mathbb{Q}\cap[0,1]} \underbrace{\{\omega \in \Omega : |B_s(\omega)-B_t(\omega)| \le C|s-t|^\alpha\}}_{\in\mathcal{A}} \in \mathcal{A}. \end{aligned}$$

(More precisely (but not necessary to point this out) we have $\{\omega \in \Omega : |B_s(\omega)-B_t(\omega)| \le C|s-t|^\alpha\} = (B_s,B_t)^{-1}(N)$ with $N := \{(y,z) \in \mathbb{R}^2 : |y-z| \le C|s-t|^\alpha\}$. Note that $N \in \mathcal{B}(\mathbb{R})^2$. The process (B_s,B_t) is $\mathcal{A}$-$\mathcal{B}(\mathbb{R})^2$-measurable since B_s, B_t are $\mathcal{A}$-$\mathcal{B}(\mathbb{R})$-measurable (result from Probability theory 1), thus $(B_s,B_t)^{-1}(N) \in \mathcal{A}$).
(ii) Since $B(\frac{k}{n}) - B(\frac{k-1}{n}) \sim N(0,\frac{1}{n})$, $k=1,...,n$ are i.i.d., we obtain with some $Z \sim N(0,1)$:

$$\mathbb{P}\big(\forall k \in \{1,...,n\} : \big|B(\frac{k}{n}) - B(\frac{k-1}{n})\big| \le Cn^{-\alpha}\big) = \mathbb{P}\big(\frac{1}{\sqrt{n}}|Z| \le Cn^{-\alpha}\big)^n = \mathbb{P}\big(|Z| \le Cn^{\frac{1}{2}-\alpha}\big)^n.$$

For n large enough, $Cn^{\frac{1}{2}-\alpha} \le 1$. In this case, the above term is smaller than $\mathbb{P}(|Z| \le 1)^n \to 0$ $(n\to\infty)$ since $\mathbb{P}(|Z| \le 1) < 1$.
(iii) If B is Hoelder continuous w.r.t. (α, C), then it would hold for all $n \in \mathbb{N}$ and for $k = 1,...,n$ that $\big|B(\frac{k}{n}) - B(\frac{k-1}{n})\big| \le C|\frac{k}{n} - \frac{k-1}{n}|^\alpha = Cn^{-\alpha}$. Thus for all $n \in \mathbb{N}$,

$$\mathbb{P}(M) \le \mathbb{P}\big(\forall k \in \{1,...,n\} : \big|B(\frac{k}{n}) - B(\frac{k-1}{n})\big| \le Cn^{-\alpha}\big) \overset{(b)}{=}\to 0.$$

This shows that $\mathbb{P}(M) = 0$, i.e. almost surely B is not Hoelder continuous w.r.t. (α, C).

(g) (i) Note that if it holds that $f(t) \le B_t \le g(t)$ for all $t \in \mathbb{Q}\cap[0,1]$, then the same holds also for $t \in [0,1]$. Proof: For $t \in [0,1]$ we can find a sequence $t_n \to t$ with $(t_n) \subset \mathbb{Q}\cap[0,1]$. By assumption, $f(t_n) \le B_{t_n} \le g(t_n)$ for all $n \in \mathbb{N}$. By $n \to \infty$, $f(t) \le B_t \le g(t)$ (*).

We have

$$\begin{aligned} M &= \{\omega \in \Omega : \forall t \in [0,1] : f(t) \le B_t(\omega) \le g(t)\} \\ &\overset{(*)}{=} \{\omega \in \Omega : \forall t \in \mathbb{Q}\cap[0,1] : f(t) \le B_t(\omega) \le g(t)\} \\ &= \bigcap_{t\in\mathbb{Q}\cap[0,1]} \underbrace{\{\omega \in \Omega : f(t) \le B_t(\omega) \le g(t)\}}_{\in\mathcal{A}} \in \mathcal{A}. \end{aligned}$$

(ii) It holds that (the inequalities are due to the fact that the conditions in the $\mathbb{P}(\cdot)$ imply the conditions in the $\mathbb{P}(\cdot)$ in the next line):

$$\begin{aligned}
&\mathbb{P}\big(\forall k \in \{0,...,n\} : \frac{k}{n^2} \leq B_{\frac{k}{n^2}} \leq 2\frac{k}{n^2}\big) \\
\leq\ &\mathbb{P}\big(\forall k \in \{0,...,n-1\} : \frac{k+1}{n^2} - \frac{2k}{n^2} \leq B_{\frac{k+1}{n^2}} - B_{\frac{k}{n^2}} \leq \frac{2(k+1)}{n^2} - \frac{k}{n^2}\big) \\
=\ &\mathbb{P}\big(\forall k \in \{0,...,n-1\} : \underbrace{\frac{-k+1}{n^2}}_{-\frac{1}{n}\leq ...} \leq \underbrace{B_{\frac{k+1}{n^2}} - B_{\frac{k}{n^2}}}_{\text{iid}\sim N(0,\frac{1}{n^2})} \leq \underbrace{\frac{k+2}{n^2}}_{...\leq\frac{n+1}{n^2}\leq\frac{2}{n}}\ \big) \\
\leq\ &\mathbb{P}\big(-\frac{1}{n} \leq N(0,\frac{1}{n^2}) \leq \frac{2}{n}\big)^n \\
\leq\ &\mathbb{P}\big(-1 \leq N(0,1) \leq 2\big)^n = a^n \to 0,
\end{aligned}$$

since $a := \mathbb{P}\big(-1 \leq N(0,1) \leq 2\big) < 1$.
(iii) If B would be surrounded by f and g, then for all $n \in \mathbb{N}$ it would hold that $\forall k \in \{0,...,n\}$: $\frac{k}{n^2} \leq B_{\frac{k}{n^2}} \leq 2\frac{k}{n^2}$. Thus by (b),

$$\mathbb{P}(M) \leq \mathbb{P}(\forall k \in \{0,...,n\} : \frac{k}{n^2} \leq B_{\frac{k}{n^2}} \leq 2\frac{k}{n^2}) \to 0.$$

This shows that $\mathbb{P}(M) = 0$, i.e. almost surely B is not surrounded by f and g.